NOTES ON CLASSICAL METHODS OF PARTIAL DIFFERENTIAL EQUATIONS

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Contents

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Notation and conventions

- Let $x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, and
$$\partial_{x_j} u = \frac{\partial u}{\partial x_j}, \quad j = 1, \dots, n.$$
The gradient of u is
$$\nabla u = (\partial_{x_1} u, \dots, \partial_{x_n} u).$$
- The Laplace operator on $\mathbb{R}^n$ is
$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$
- The d'Alembert operator on $\mathbb{R}^n$ is
$$\Box u = \partial_{x_1}^2 u - \Delta_{\mathbb{R}^{n-1}} u = \partial_{x_1}^2 u - \sum_{j=2}^{n} \partial_{x_j}^2 u.$$
- A vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is called a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We denote
$$\nabla^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$
and similarly for $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
- $\overline{\Omega}$ is the closure of Ω, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ is the boundary.
- For Ω bounded, we equip this set with the norm:
$$\|u\|_{C^m(\Omega)} = \sum_{|\alpha| \le m} \sup_{\overline{\Omega}} \|\partial^\alpha u\|_{L^\infty}.$$
This yields a Banach space.
- $C^m(\Omega)$ is the space of m-times differentiable functions, which are continuous up to $\partial\Omega$.
- $C^\infty(\Omega) = \bigcap_{m \in \mathbb{N}} C^m(\Omega)$ is the space of smooth functions.

Introduction

These lecture notes are based on Math 576 taught at the University of Illinois Chicago by Prof. Christof Sparber in the Spring 2025 semester and by Prof. Irina Nenciu in the Spring 2024 semester.

1. The Basics

1.1. What is a PDE?.

Definition 1.1. A *partial differential equation (PDE)* is an equation for an unknown $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^k$ and its derivatives, where $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ is some open connected subset of $\mathbb{R}^n$. A scalar PDE is obtained for $k = 1$.

Definition 1.2. The support of u is

$$\operatorname{supp} u = \overline{\{x \in \mathbb{R}^n : u(x) \neq 0\}}.$$

Thus,

$$C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) : \operatorname{supp} u \text{ is a compact subset of } \mathbb{R}^n\}.$$

Definition 1.3. Let $\alpha, \beta, \dots$ be multi-indices, and let

$$F : \Omega \times \mathbb{R}^k \to \mathbb{R}$$

such that

$$L(u) = F\left(x, \partial^\alpha u, \partial^\beta u, \dots\right) = 0.$$

We call this a PDE of order $m = \max(|\alpha|, |\beta|, \dots)$.

- A PDE is *linear* if F if is linear with respect to the derivatives of u.
- A PDE is *quasi-linear* if F is linear with respect to the highest-order derivatives (of order m):

$$F(x, \partial^\alpha u, \dots) = \sum_{|\alpha|=m} a_\alpha(x, u, \partial^\beta u)\partial^\alpha u + f(x, \partial^\beta u, u),$$

  where

$$L_p(u) = \text{the principal part of the differential operator.}$$

Definition 1.4. A PDE problem is $Lu = 0$ together with possible initial/boundary states for u.

Question 1.5. Some important questions we might ask are:

- Existence of a solution u?
- Uniequness of the solution?
- Continuous dependence on the data?

Definition 1.6. (Hadamard) A PDE problem is *well-posed* if there exists a unique solution u depending continuously on the data.

1.2. The Cauchy Problem.

For $u : \mathbb{R}^n \to \mathbb{R}$, we consider a quasi-linear PDE of order $m \in \mathbb{N}$, i.e.,

$$L(u) := \sum_{|\alpha|=m} a_\alpha(x, u)\nabla^\alpha u + f(x, u) = 0, \quad x \in \Omega \subset \mathbb{R}^n(*) \tag{1.1}$$

where a_α and f involve $\nabla^\alpha u$ up to order $|\alpha| \leq m - 1$.

Definition 1.7. Let Γ be a C^1-hypersurface in $\Omega \subset \mathbb{R}^n$ with associated unit normal vector ν. Then, its directional derivative in direction ν is

$$\frac{\partial u}{\partial \nu} = \nu \cdot \nabla u.$$

Definition 1.8. Let $g_1, \dots, g_{m-1} : \Gamma \to \mathbb{R}$ be some given functions. A (unique) solution u to 1.1 such that
$$\frac{\partial^j u}{\partial \nu^j} = g_j \quad \text{on } \Gamma$$
is called the **Cauchy problem**.

In the following, we assume that the data g_j are analytic. Denote
$$S = \Gamma \cup \{g_1, \dots, g_{m-1}\}.$$

Definition 1.9. S is called *characteristic* for L at $x_0 \in \Gamma$ if $L(u)$ at x_0 cannot be determined from the Cauchy data. If this holds $\forall x_0 \in \Gamma$, we say S is characteristic for L.

Remark 1.10. For linear PDEs, S is characteristic independently of the Cauchy data. Thus, Γ is called a characteristic.

Theorem 1.11. *Denote by*
$$L_p(\xi, u) = \sum_{|\alpha|=m} a_\alpha(x,u)\xi^\alpha, \quad \xi \in \mathbb{R}$$
the principal symbol of L. Then S is characteristic at $x_0 \in \Gamma$ if and only if
$$L_p\big(\nu, u(x_0)\big) = 0,$$
where ν is the unit normal vector on Γ.

Proof. For C^1-hypersurfaces Γ, there exists a local differentiable coordinate transformation $x \mapsto \gamma$ mapping a neighborhood of x_0 such that the image of Γ is given by a level set $\gamma_1 = 0$, with $\nabla\gamma_1$ orthogonal to Γ and $\gamma_2, \dots, \gamma_n$ spanning the tangent space.

The inverse map $\gamma \mapsto x$ yields:
$$x = x_\Gamma(\gamma_2, \dots, \gamma_n) + \widetilde{\nu}(\gamma_2, \dots, \gamma_n)\gamma_1.$$

Thus, all normal derivatives $(\nabla\gamma_1)^\alpha$ for $|\alpha| \le m-1$ are given through the Cauchy data.

Changing coordinates in the PDE, we obtain
$$L(u) = \sum_{|\alpha|=m} a_\alpha(x_0,u)\nabla^\alpha u + f(x,u)$$
$$= \sum_{|\alpha|=m} a_\alpha(x_0,u)\frac{\partial^m u}{\partial \gamma_1^m}(\nabla\gamma_1)^\alpha + g(x_0,u)$$
$$= \frac{\partial^m u}{\partial \gamma_1^m} L_p\big(\nabla\gamma_1, u(x_0)\big) + g(x,u).$$
Since the higher order normal derivatives are not obtained from the Cauchy data, for Σ to be characteristic, we must have:
$$L_p\big(\nabla\gamma_1, u(x_0)\big) = 0.$$
□

Example 1.12. For $(x,y) \in \mathbb{R}^2$, consider the Cauchy problem
$$\begin{cases} L(u) = \frac{\partial^2 u}{\partial x \partial y} = 0, \\ u(x,0) = g_0(x), \quad \partial_y u(x,0) = g_1(x), \end{cases}$$
where g_0, g_1 are analytic.

Since $\Gamma = \{(x,0) \in \mathbb{R}^2\}$, we have $\nu = (0,1)$, and for $\xi = (\xi_1, \xi_2)$, the principal symbol is
$$L_p(\xi) = \xi_1\xi_2.$$

Then $L_p(\xi) = 0$ on Γ, it is characteristic.

However, we might face the following problems.

- The value of $L(u)$ on Γ satisfies
$$\frac{\partial^2 u}{\partial x \partial y}(x,0) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}(x,0)\right) = g_1'(x).$$
If $g_1' \neq 0$, this leads to a contradiction with $\frac{\partial^2 u}{\partial x \partial y} = 0$, implying that no analytic solution u will exist.
- If $g_1 = 0$, i.e., $g_1(x) = k$ (a constant), then one can check that
$$u(x,y) = g(x) + ky + f(y)$$
solves the Cauchy problem for all f such that $f(0) = f'(0) = 0$. Thus, there is no uniqueness in this case.

Example 1.13. For $(x,y) \in \mathbb{R}^2$, u, g_0, g_1 analytic consider:
$$\begin{cases} \Box u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \\ u(x,0) = g_0(x), \quad \partial_y u(x,0) = g_1(x). \end{cases}$$
So $\Gamma = \{y = 0\} \subset \mathbb{R}^2$, then we have
$$\nu = (0,1), L_p(\xi) = \xi_1^2 - \xi_2^2,$$
hence $L_p(\nu) = -1 \neq 0$ not characteristic!

Claim: we can find all derivatives of u on Γ. Denote
$$D^{(k,l)}u = \frac{\partial^{k+l}}{\partial x^k \partial y^l} u.$$

Then,
$$D^{(k,0)}u(x,0) = g_0^{(k)}(x) \quad \forall k \geq 1.$$
$$D^{(k,1)}u(x,0) = g_1^{(k)}(x) \quad \forall k \geq 1.$$
For $l \geq 2$ and k, using the PDE, we get

$$\begin{gathered} D^{(k,l)}u(x,0) \\ D^{(k,l-2)}\frac{\partial^2 u}{\partial y^2}(x,0) \\ = D^{(k,l-2)}\frac{\partial^2 u}{\partial x^2}(x,0) \\ D^{(k+2,l-2)}u(x,0). \end{gathered}$$

Thus, this gives a recursive relationship for all derivatives, requiring only $D^{(k,0)}u$ and $D^{(k,1)}u$ on Γ. One finds:
$$D^{(k,l)}u(x,0) = \begin{cases} g_0^{(k+l)}(x), & l \text{ even}, \\ g_1^{(k+l-1)}(x), & l \text{ odd}. \end{cases}$$

Expanding u in a Taylor series near $y = 0$ yields
$$u(x,y) = \sum_{l=0}^{\infty} \frac{D^{(0,l)}u(x,0)}{l!} y^l.$$
Since
$$D^{(0,2l)}u(x,0) = g^{(2l)}(x), \quad D^{(0,2l+1)}u(x,0) = g^{(2l+1)}(x),$$
we obtain the unique analytic solution.

Example 1.14. For $(x,y) \in \mathbb{R}^2$ and $\Gamma = \{y=0\} \subset \mathbb{R}^2$. we consider

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = u^2 \\ u(x,0) = 1, \quad \frac{\partial u}{\partial y}(x,0) = x. \end{cases}$$

Since

$$L_p(\xi, u) = \xi_1^2 + u\xi_2^2,$$

and $\nu = (0,1)$,

$$L_p(\nu, u(x,0)) = 0 + u(x,0) = 1$$

is not characteristic. Notice that we can find all derivatives of u on Γ:

$$D^{(k,0)}u(x,0) = 0, \quad \forall k \geq 1.$$

$$D^{(k,1)}u(x,0) = \begin{cases} 1, k=1 \\ 0, k \geq 2 \end{cases}$$

For $l = 2$, rewriting the PDE generates

$$\frac{\partial^2 u}{\partial y^2}(x,0) = u(x,0) + \frac{1}{u(x,0)}\left(\frac{\partial u}{\partial y}(x,0) - \frac{\partial^2 u}{\partial x^2}(x,0)\right) = 1 + x.$$

Taking more derivatives yields

$$D^{(k,2)}u(x,0) = \frac{\partial^k}{\partial x^k}(1+x) = \begin{cases} 1, k=1 \\ 0, k \geq 2 \end{cases}$$

More generally, $D^{(k,l)}u(x,0)$ for $l > 2$ is obtained by differentiating the PDE.

Theorem 1.15 (Cauchy-Kovalevskaya)**.** *Let L be a quasilinear differential operator with analytic coefficients. Assume the initial surface $S = \Gamma \cup \{g_0, \ldots, g_{m-1}\}$ is analytic and non-characteristic for L. Then the Cauchy problem*

$$\begin{cases} L(u) = 0, \\ u|_\Gamma = g_0, \end{cases}$$

has a unique analytic solution locally near Γ.

Proof. The proof follows by expanding u in a Taylor series (possibly after a suitable change of variables to "flatten" Γ)and determining the coefficients using the Cauchy data and PDE. □

Remark 1.16. In practice, this theorem is not very useful because:

- It only provides a local existence result in general.
- The analyticity assumptions are often too strong.

In addition, one faces other issues, as the following example shows.

Example 1.17. For $(x,y) \in \mathbb{R}^2$, consider

$$\begin{cases} \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u(x,0) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right) \\ \frac{\partial u}{\partial y}(x,0) = 0 \end{cases} \qquad \text{for some } \varepsilon > 0.$$

This is a non-characteristic Cauchy problem with analytic data. One checks that the unique solution is given by

$$u(x,y) = \frac{\varepsilon}{2}\sin\left(\frac{x}{\varepsilon}\right)e^{\frac{y}{\varepsilon}} + \frac{\varepsilon}{2}\sin\left(\frac{x}{\varepsilon}\right)e^{-\frac{y}{\varepsilon}}.$$

Taking the limit as $\varepsilon \to 0$, we see that the solution does not converge to a single analytic function.

Remark 1.18. Cauchy problems for the Laplace equation (or other elliptic PDEs) are in general ill-posed, while boundary-value problems are usually well-posed.

1.3. **The method of characteristics for first order (quasi-linear) PDEs.** For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we consider equations of the form:

$$\sum_{j=1}^{n} a_j(x, u)\partial_{x_j} u = f(x, u) \tag{1.2}$$

with C^1-coefficients a_j and f. Then the principal part can be calculated as

$$\begin{aligned} L_p(x, u) &= \sum_{j=1}^{n} a_j(x, u)\xi \\ &= \vec{a}(x, u) \cdot \vec{\xi}. \end{aligned}$$

Our goal is to turn the PDE into system of ODE. Let $s \in I = (s_0, s_1) \subset \mathbb{R}$ and define curves

$$\begin{aligned} x : I &\to \mathbb{R}^n \\ s &\mapsto x(s) \end{aligned}$$

such that

$$\dot{x}_j(s) = a_j\Big(\, x(s), u\big(\, x(s)\,\big)\,\Big)$$

and

$$z(s) = u\big(\, x(s)\,\big).$$

Then the L.H.S. of 1.2 is equal to

$$\dot{z}(s) = \sum_{j=1}^{n} \frac{du\big(\, x(s)\,\big)}{\partial x_j} \underbrace{\dot{x}_j(s)}_{=a_j(x(s), u(s))} .$$

Therefore, in view of $(*)$, we get

$$\begin{cases} \dot{x}(s) = a(x(s), z(s)) \\ \dot{z}(s) = f(x(s), z(s)) \end{cases}$$

which are the characteristic ODEs associated to 1.2 . More precisely, we can use this idea to solve the Cauchy problem associated to 1.2, where without loss of generality, we consider

$$\Gamma = \{x \in \mathbb{R}^n : x_n = 0\},$$

hence $v = (0, 0 \cdots, 0, 1)$, giving us the system

$$\begin{cases} \sum_{j=1}^{n} a^j(x_j, u)\, \partial x_j u = f(x, u), \\ u|_\Gamma = u_0(y), \quad y \in \mathbb{R}^{n-1}. \end{cases} \tag{1.3}$$

Theorem 1.19 (Local Solvability)**.** *Let $U \subset \mathbb{R}^{n-1}$ be open, and let $y_0 \in U$. Suppose*

$$f,\, \{a_j\}_{j \in \mathbb{N}} \in C^1(\mathbb{R}^n \times \mathbb{R}),\; u_0 \in C^1(U),$$

and assume that

$$S = \Gamma \cup \{u_0\}$$

is non-characteristic at $(y_0, 0)$; i.e.,

$$\vec{a}\big((y_0, 0),\, u(y_0)\big)\vec{\nu} \equiv a_n\big((y_0, 0),\, u_0(y_0)\big) \neq 0.$$

Then there exists an open set $W \subset \mathbb{R}^n$ with $W \cap (\mathbb{R}^{n-1} \times 0) \subset U$ such that 1.3 has a unique solution $u \in C^1(W)$ with $u|_\Gamma = u(y, 0) = u_0(y)$.

Proof. The solution u is obtained from the characteristic ODEs with initial data

$$x(0,y) = (y,0) \quad \text{and} \quad z(0,y) = u_0(y), \quad \text{i.e.,}$$

$$\begin{cases} \dot{x}(s,y) = a\big(\,x(s,y), z(s,y)\,\big), & x(0,y) = (y,0), \\ \dot{z}(s,y) = f\big(\,x(s,y), z(s,y)\,\big), & z(0,y) = u_0(y). \end{cases}$$

Since a and f are in C^1, the theory of ODEs implies that for every $y \in \mathbb{R}^{n-1}$, there exists a maximal solution $\big(\,x(s,y), z(s,y)\,\big)$ defined for $s \in I_y \subset \mathbb{R}$ containing the origin (by the Cauchy–Lipschitz theorem).

The function

$$(s,y) \mapsto x(s,y)$$

is C^1 on an open set $\cup_{y\in U}\{y\} \times I_y$. We want to define uniquely

$$u\big(\,x(s,y)\,\big) = z(s,y)$$

for $s \neq 0$. This requires that the map

$$y \mapsto x(s,y)$$

needs to be injective (i.e., characteristics don't cross).

Moreover, as s and y vary, the curves $x(s,y)$ should trace out an open set $W \subset \mathbb{R}^n$. Both of these follow from the inverse function theorem if the Jacobian of

$$(y,s) \mapsto x(s,y)$$

is non-zero at $s=0$ and $y = y_0$. Since $x(0,y) = (y,0) \in \mathbb{R}^n$,

$$\partial_{y_j} x_i(0,y) = \delta_{ij}, \quad \partial_s x_i(0,y) = 0 \quad \text{if } i \neq n.$$

We also have

$$\partial_s x(0,y) = a((y,0),u_0(y))$$

in view of 1.2. Hence, the Jacobian:

$$\det\left(\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_{n-1}} & \frac{\partial x_1}{\partial s} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_{n-1}} & \frac{\partial x_2}{\partial s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_{n-1}} & \frac{\partial x_n}{\partial s} \end{bmatrix}\right) = \begin{vmatrix} 1 & 0 & 0 & \cdots & a_1((y_0,0),u_0(y_0)) \\ 0 & 1 & 0 & \cdots & a_2((y_0,0),u_0(y_0)) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n((y_0,0),u_0(y_0)) \end{vmatrix}.$$

This simplifies to

$$a_n((y_0,0),u_0(y_0)) \cdot v$$

by the non-characteristic condition. Hence,

$$a((y_0,0),u_0(y_0)) \neq 0.$$

□

Example 1.20. Consider

$$\begin{cases} u\,\partial_{x_1} u + x_2\,\partial_{x_2} u = x_1, \\ u(x_1,1) = 2x_1. \end{cases}$$

Hence,

$$\Gamma = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = y, x_2 = 1\} = (y,1) \in \mathbb{R}^2$$

is a parametrization of Γ. Using the method of characteristics, we get

$$\begin{cases} \dot{x}_1(s;y) = z(s;y), & x_1(0;y) = y, \\ \dot{x}_2(s;y) = x_2(s;y), & x_2(0;y) = 1, \\ \dot{z}(s;y) = x_1(s;y), & z(0;y) = 2y. \end{cases}$$

The Jacobian is given by:

$$\det \begin{pmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial s} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial s} \end{pmatrix} = \det \begin{pmatrix} 1 & z(0,y) \\ 0 & x_2(0,y) \end{pmatrix}.$$

This simplifies to

$$\det \begin{pmatrix} 1 & 2y \\ 0 & 1 \end{pmatrix} = 1 \neq 0,$$

so Γ is non-characteristic. We proceed by solving the system of ODEs. Clearly,

$$x_2(s;y) = C_0(y)e^s.$$

$C_0 = 1$ from $x_2(0;y) = 1$. Notice

$$\ddot{x}_1(s;y) = \dot{z}(s;y) = x_1(s;y).$$

Equivalently,

$$\ddot{x}_1(s;y) - x_1(s;y) = 0.$$

Using the classic result from ODE, we get

$$\begin{cases} x_1(s,y) = C_1(y)e^s + C_2(y)e^{-s} \\ z(s,y) = \dot{x}_1(s,y) = C_1(y)e^s - C_2(y)e^{-s} \end{cases}$$

Applying the conditions at $s = 0$, the solution is

$$\begin{cases} x_1(s,y) = \frac{3}{2}ye^s - \frac{y}{2}e^{-s} \\ z(s,y) = \dot{x}_1(s,y) = \frac{3}{2}ye^s + \frac{y}{2}e^{-s}. \end{cases}$$

To invert $(s,y) \mapsto (x_1, x_2)$, we subsitute $x_2 = e^s$ into x_1:

$$x_1 = \frac{3}{2}yx_2 - \frac{y}{2x_2}.$$

yielding

$$y = \frac{2x_1x_2}{3x_2^2 - 1}.$$

Hence

$$u(x_1,x_2) = z\big(\, s(x_1,x_2), y(x_1,x_2)\,\big)$$

$$= \frac{2x_1x_2}{3x_2^2-1}\left(\frac{3x_2}{2} + \frac{1}{2x_2}\right) = \frac{x_1(3x_2^2+1)}{3x_2^2-1},$$

for $x_2 \neq \pm\frac{1}{\sqrt{3}}$. One checks that the Jacobian

$$\det \begin{pmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial y} \\ \frac{\partial x_2}{\partial s} & \frac{\partial x_2}{\partial y} \end{pmatrix}(s,y) \neq 0 \quad \text{if and only if} \quad x_2 = e^s \neq \frac{1}{\sqrt{3}}.$$

Remark: This method can be generalized to fully nonlinear, first-order PDEs as shown in [2].

1.4. **Classification of second order PDEs.**

Definition 1.21. A general second-order differential equation (quasi-linear) can be written as

$$L(u)=\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+f(x,u,\nabla u)=0 \tag{1.4}$$

where all $a_{ij}=a_{ij}(x,u,\nabla u)$. We can without loss of generality assume that $A=(a_{ij})_{i,j=1}^n$ is symmetric, since otherwise

$$a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+a_{ji}\frac{\partial^2 u}{\partial x_j \partial x_i}=(a_{ij}+a_{ji})\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

$$=\frac{1}{2}(a_{ij}+a_{ji})\frac{\partial^2 u}{\partial x_i \partial x_j}+\frac{1}{2}(a_{ij}+a_{ji})\frac{\partial^2 u}{\partial x_j \partial x_i}.$$

where $(b_{ij})_{i,j=1}^n=\frac{1}{2}(a_{ij}+a_{ji})$ is symmetric.

Definition 1.22. Let $\lambda_1,\dots,\lambda_k\in\mathbb{R}$ be the eigenvalues of A.

(a) We call 1.4 **elliptic** at x, if $\lambda_j>0$ for all $j=1,\dots,n$.

(b) We call 1.4 **hyperbolic** at x, if $\exists j\in\{1,\dots,n\}$ s.t. $\lambda_j>0$ and $\lambda_i<0$ for $i\neq j$. (or vice versa)

(c) We call 1.4 **parabolic** if $\exists j\in\{1,\dots,n\}$ s.t. $\lambda_j=0$ and all other λ_i $(i\neq j)$ have the same sign (positive or negative).

Remark 1.23. For $n=2$, we have $A=\begin{pmatrix} a & b\\ b & c\end{pmatrix}$ such that $L(u)=\operatorname{div}(A\nabla u)$. The eigenvalues are

$$\lambda_{1,2}=\frac{1}{2}(a+c)\pm\frac{1}{2}\sqrt{(a+c)^2+4(b^2-ac)}\ \in\mathbb{R}$$

Hence, we conclude

$$\lambda_{1,2}>0 \iff b^2-ac<0.$$

$$\lambda_1>0,\ \lambda_2<0 \iff b^2-ac>0.$$

$$\lambda_1=0 \iff b^2-ac=0.$$

Example 1.24. For the Laplace equation on $\mathbb{R}^n$, $\Delta u=0$, we have A equals the identity matrix and all its eigenvalues equal to 1. Hence, the equation is elliptic.

Example 1.25. Consider the wave equation for $(t,x_1,\dots,x_n)\in\mathbb{R}^{n+1}$:

$$\partial_{tt}-\Delta u=0.$$

Then

$$A=\begin{vmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & -1 & 0 & \cdots & 0\\ 0 & 0 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -1\end{vmatrix}$$

which ahs one eigenvalue equals to 1 and $n-1$ eigenvalues equal to -1. Thus, it is hyperbolic.

Example 1.26. Consider the heat equation on $\mathbb{R}^{n+1}$:

$$\partial_t u-\Delta u=0.$$

Then

$$A = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix},$$

and hence the equation is parabolic.

Note that the initial value problem:

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u|_{t=0} = u_0(x), \ x \in \mathbb{R}^n \end{cases}$$

is not a Cauchy problem, but usually still well-posed.

2. Distribution Theory

2.1. Basic notions from real analysis.

Definition 2.1. Let $1 \le p < \infty$, we denote, for Ω open and $\Omega \neq \emptyset$,

$$L^p(\Omega) = \{f : \Omega \to \mathbb{C} \text{ measurable and } \|f\|_{L^p} < \infty\},$$

where

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}.$$

Proposition 2.2. *It holds that* $L^p(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{L^p}}$, *i.e.,* C_0^∞ *is dense in* $L^p(\Omega)$.

Definition 2.3. For $p = \infty$, we define:

$$\|f\|_{L^\infty} = \text{ess sup}_{x\in\Omega}|f(x)| = \inf\{C > 0 : |f(x)| \le C \text{ for almost all } x \in \Omega\}.$$

Proposition 2.4. *The spaces* $L^p(\Omega)$ *with* $1 \le p \le \infty$ *are Banach spaces, i.e., complete.*

Definition 2.5. For $p = 2$, we obtain the Hilbert space $L^2(\Omega)$ with the inner product:

$$\langle f, g\rangle = \int_\Omega f\overline{g} \, dx,$$

such that $\|f\|_{L^2}^2 = \langle f, f\rangle$. We say that f is orthogonal to g in $L^2(\Omega)$ if $\langle f, g\rangle = 0$.

Lemma 2.6 (Hölder inequality)**.** *Let* $1 \le p, q \le \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for all* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$*:*

$$\left|\int_\Omega f\overline{g} \, dx\right| \le \|f\|_{L^p}\|g\|_{L^q}.$$

For $p = q = 2$, *we obtain:*

$$|\langle f, g\rangle| \le \|f\|_{L^2}\|g\|_{L^2},$$

which is the Cauchy-Schwarz inequality.

Theorem 2.7 (Lebesgue dominated convergence)**.** *Let* $\{f_j\}_{j\in\mathbb{N}}$ *be a sequence of measurable functions such that* $\lim_{j\to\infty} f_j(x) = f(x)$ *almost everywhere on* $\Omega \subset \mathbb{R}^n$. *If there exists* $g \in L^1(\Omega)$ *such that* $|f_j(x)| \le g(x)$ *for all* $j \ge 1$, $x \in \Omega$, *then*

$$\lim_{j\to\infty} \int_\Omega f_j \, dx = \int_\Omega f \, dx.$$

Remark 2.8. If $\text{meas}(\Omega) < \infty$, i.e., Ω is a bounded domain, then $g(x) = \text{constant}$ is sufficient.

2.2. Distributions on $\mathcal{D}'(\Omega)$.

Definition 2.9. For $\Omega \subset \mathbb{R}^n$ open and $\Omega \neq \emptyset$, we call

$$\mathcal{D}(\Omega) := C_0^\infty(\Omega) = \{f : \Omega \to \mathbb{C}, f \text{ smooth with compact support}\},$$

the space of test functions.

Definition 2.10. A sequence $\{\phi_m\}_{m\ge1}$ of test functions is said to converge to ϕ in $\mathcal{D}(\Omega)$ if there exists a compact $K \subset \Omega$ such that $\text{supp } \phi_m \subset K$ for all $m \ge 1$, and

$$\sup_{x\in K} |\nabla_x^\alpha \phi_m - \nabla_x^\alpha \phi| \to 0, \quad \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n.$$

Example 2.11.

$$\phi(x) = \begin{cases} 0, & |x| \ge R > 0, \\ \exp\left(\frac{R^2}{|x|^2 - R^2}\right), & |x| < R, \end{cases}$$

is in $\mathcal{D}(\Omega)$ (Homework).

Definition 2.12. A distribution $u \in \mathcal{D}'(\Omega)$ is a linear continuous functional $u : \mathcal{D}(\Omega) \to \mathbb{C}$, $\varphi \mapsto u(\varphi)$, i.e.,

- $u(\alpha\varphi_1 + \beta\varphi_2) = \alpha u(\varphi_1) + \beta u(\varphi_2)$, $\forall \alpha, \beta \in \mathbb{C}$,
- If $\varphi_m \to \varphi$ in $\mathcal{D}(\Omega)$, then $u(\varphi_m) \to u(\varphi)$.

Remark 2.13. $\mathcal{D}'(\Omega)$ is called the dual space of $\mathcal{D}(\Omega)$, and we say that $\{u_j\}_{j\geq 1} \subset \mathcal{D}'(\Omega)$ converges $u_j \to u$ in $D'(\Omega)$ iff:

$$\lim_{j\to\infty} u_j(\varphi) = u(\varphi), \quad \forall \varphi \in D(\Omega).$$

Lemma 2.14 (Regular distributions)**.** *Define*

$$L^\infty_{loc}(\Omega) = \{f : \Omega \to \mathbb{C} \mid \int_K |f(x)| dx < \infty, \forall \textit{ compact } K \subset \Omega\}.$$

Then any $f \in L^\infty_{loc}(\Omega)$ defines a regular distribution $u_f \in \mathcal{D}'(\Omega)$ via:

$$u_f(\varphi) = \int_\Omega f(x)\varphi(x)\, dx, \quad \forall \varphi \in D(\Omega).$$

Proof. Clearly, u_f is linear and well-defined since:

$$|u_f(\varphi)| \leq \int_K |f||\varphi| dx \leq \|\varphi\|_{L^\infty} \int_K |f|\, dx < \infty,$$

by Hölder's inequality. The Lebesgue DCT yields continuity (exercise). □

Notation: From now on, we write $u(\varphi) = \langle u, \varphi\rangle$, the duality bracket. For $f \in L^\infty_{\mathrm{loc}}(\Omega)$, we write, by abuse of notation,

$$u_f(\varphi) = \langle f, \varphi\rangle.$$

Definition 2.15 (Dirac Distribution)**.** Given $x_0 \in \Omega$, we define $\delta_{x_0} : \mathcal{D}(\Omega) \to \mathbb{C}$ via

$$\langle \delta_{x_0}, \varphi\rangle = \varphi(x_0) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Theorem 2.16. *The δ_{x_0}-distribution is not a regular distribution but can be obtained as the limit of such a regular distribution.*

Example 2.17. Consider the funtion

$$f_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon} & \text{if } |x - x_0| \leq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $f_\epsilon \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} |f_\epsilon| dx = 1\ \forall \epsilon > 0$. We claim that $f_\epsilon \to \delta_{x_0}$ in $\mathcal{D}'(\mathbb{R})$, i.e.,

$$\lim_{\epsilon\to 0} \langle f_\epsilon, \varphi\rangle = \langle \delta_{x_0}, \varphi\rangle = \varphi(x_0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

To see this let W.L.O.G. $x_0 = 0$ Then,

$$\langle f_\epsilon, \varphi\rangle = \frac{1}{2\epsilon} \int_{|x|\leq\epsilon} \varphi(x)\, dx.$$

$$= \frac{1}{2\epsilon} \left(\int_{|x|\leq\epsilon} \big(\varphi(x) - \varphi(0)\big) dx + \varphi(0) \int_{|x|\leq\epsilon} dx \right)$$
$$= \frac{1}{2\epsilon} \int_{|x|\leq\epsilon} \big(\varphi(x) - \varphi(0)\big) dx + \varphi(0) \int_{|x|\leq\epsilon} \frac{1}{2\epsilon} dx.$$

Here, the second integral $\int_{|x|\leq\epsilon} \frac{1}{2\epsilon} dx = 1$ for all $\epsilon > 0$. For the first integral, subsitute $y = \frac{x}{\epsilon}$ yields

$$\langle f_\epsilon, \varphi\rangle = \frac{1}{2} \int_{-1}^{1} \big(\varphi(\epsilon y) - \varphi(0)\big) dy + \varphi(0).$$

by the Dominated Convergence Theorem,

$$\lim_{\epsilon\to 0}\frac{1}{2}\int_{-1}^{1}\big(\,\varphi(\epsilon y)-\varphi(0)\,\big)dy=0.$$

Thus,

$$\lim_{\epsilon\to 0}\langle f_\epsilon,\varphi\rangle=\varphi(0)=\langle\delta_0,\varphi\rangle,\quad \forall\varphi\in\mathcal{D}(\mathbb{R}).$$

Definition 2.18. Let O be an open subset of $\Omega\subseteq\mathbb{R}^n$, and let $u\in\mathcal{D}'(\Omega)$. We say u vanishes on O if

$$\langle u,\varphi\rangle=0\quad\forall\varphi\in\mathcal{D}(O),\text{ where }\operatorname{supp}(\varphi)\subset O.$$

Let $\widetilde{O}$ be the maximal open set in Ω on which u vanishes. Then, $\Omega\backslash\widetilde{O}$ is called the support of $u\in\mathcal{D}'(\Omega)$.

Example 2.19. δ_{x_0} vanishes on any $O\subseteq\mathbb{R}^n$ not containing $x_0\in\mathbb{R}^n$. Thus,

$$\operatorname{supp}(\delta_{x_0})=\{x_0\}.$$

Theorem 2.20. *Let $L:\mathcal{D}(\Omega_1)\to\mathcal{D}(\Omega_2)$ be a linear continuous map. Assume L has an adjoint $L^*:\mathcal{D}(\Omega_2)\to\mathcal{D}(\Omega_1)$ such that*

$$\langle\varphi,L^*(\psi)\rangle=\int_{\Omega_1}\varphi(x)L^*(\psi)(x)\,dx=\int_{\Omega_2}L(\varphi)(y)\psi(y)\,dy=\langle L(\varphi),\psi\rangle,$$

for all $\varphi\in\mathcal{D}(\Omega_1)$, $\psi\in\mathcal{D}(\Omega_2)$.

Then, L can be continuously extended to $L:\mathcal{D}'(\Omega_1)\to\mathcal{D}'(\Omega_2)$ via

$$\langle L(u),\psi\rangle=\langle u,L^*(\psi)\rangle,\quad\forall u\in\mathcal{D}'(\Omega_1),\psi\in\mathcal{D}(\Omega_2).$$

Proof. L is linear since L^* is linear by assumption and $\langle\cdot,\cdot\rangle$ is linear in its first argument. Continuity follows from the convergence $u_j\to u$ in $\mathcal{D}'(\Omega_1)$. □

Definition 2.21. Let $a\in C^\infty(\Omega)$ and $\varphi\in\mathcal{D}(\Omega)$. Define

$$M_a(\varphi)(x)=a(x)\varphi(x).$$

Then $M_a^*=M_a$. Hence, multiplication of $u\in\mathcal{D}'(\Omega)$ by a is given by

$$\langle M_a(u),\varphi\rangle=\langle u,a\varphi\rangle,\quad\forall\varphi\in\mathcal{D}(\Omega).$$

Definition 2.22. Let $\eta:\Omega_2\to\Omega_1$ be a C^∞-diffeomorphism. For $\varphi\in\mathcal{D}(\Omega_1)$, let $L(\varphi)=\varphi\circ\eta$. Then the transformation formula is

$$\int_{\Omega_2}\varphi\big(\,(\eta(y)\,\big)\psi(y)dy=\int_{\Omega_1}\varphi(x)\psi\big(\,\eta^{-1}(x)\,\big)|\det D\eta^{-1}|dy$$

Thus,

$$\langle u\circ\eta,\psi\rangle=\langle u,\psi\circ\eta^{-1}|\det D\eta^{-1}|\rangle.$$

for all $u\in\mathcal{D}'(\Omega_1),\psi\in\mathcal{D}(\Omega_2)$.

Example 2.23 (Translations and reflections)**.** Consider the following

(a) Translations: Let $y\in\mathbb{R}^n$ and define

$$\tau_y(\phi)\;=\;\varphi(x-y)\quad\text{for}\quad\varphi\in D(\mathbb{R}^n).$$

Then $\tau_y^{-1}\;=\;\tau_{-y}$, and hence

$$\delta_y\;=\;\delta_0\circ\tau_y.$$

is defined and

$$\langle\delta_y,\psi\rangle\;=\;\langle\delta_0\circ\tau_y,\psi\rangle=\langle\delta_0,\psi\circ\tau_y^{-1}\rangle\;=\;\psi(y).$$

(b) Reflections: Let $R(\varphi)(x) = \varphi(-x)$ for $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then $R^* = R$, hence for all $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle R\,u, \psi\rangle \;=\; \langle u, R\,\psi\rangle.$$

For $\varphi, \psi \in \mathcal{D}(\Omega)$, we compute$\langle \nabla^\alpha \varphi, \psi\rangle = (-1)^{|\alpha|}\langle \varphi, \nabla^\alpha \psi\rangle$ where boundary terms vanish since φ, ψ have compact support $\subset \Omega$.

Definition 2.24. The derivative of a distribution $u \in \mathcal{D}'(\Omega)$ is given by:

$$\langle \nabla^\alpha u, \psi\rangle = (-1)^{|\alpha|}\langle u, \nabla^\alpha \psi\rangle, \quad \forall \psi \in \mathcal{D}(\Omega), \quad \forall \alpha \in \mathbb{N}_0.$$

Example 2.25. Let $H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0 \end{cases}$ be the Heaviside function.

Then $H \in L^1_{loc}(\mathbb{R})$ and its derivative in $\mathcal{D}'(\mathbb{R})$ satisfies:

$$\langle H', \psi\rangle = -\langle H, \psi'\rangle = -\int_0^\infty \psi'(x)\,dx = \psi(0) - \psi(\infty) = \psi(0),$$

since $\psi \in \mathcal{D}(\mathbb{R})$. Thus,

$$\langle H', \psi\rangle = \psi(0) \quad \forall \psi \in \mathcal{D}(\mathbb{R}),$$

i.e. $H' = \delta_0$ in $\mathcal{D}'(\mathbb{R})$.

Remark 2.26. Using this definition, we see that every distribution is infinitely often differentiable.

Example 2.27.

$$\langle \nabla^\alpha \delta_0, \psi\rangle = (-1)^{|\alpha|}\langle \delta_0, \nabla^\alpha \psi\rangle = (-1)^{|\alpha|}\nabla^\alpha \psi(0).$$

Definition 2.28. Let $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$. Their convolution $\varphi * \psi \in \mathcal{D}(\mathbb{R}^n)$ is given by:

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x-y)\psi(y)\,dy$$

$$= \int_{\mathbb{R}^n} \varphi(y)\psi(x-y)\,dy = (\psi * \varphi)(x).$$

One checks (homework) that:

$$\nabla^\alpha(\varphi * \psi) = \nabla^\alpha \varphi * \psi = \varphi * \nabla^\alpha \psi, \quad \forall \varphi, \psi \in \mathcal{D}(\mathbb{R}^n).$$

Let $f, \varphi, \psi \in \mathcal{D}'(\mathbb{R}^n)$ and check $L_f(\varphi) = f * \varphi$. Then:

$$\langle L_f(\varphi), \psi\rangle = \langle f * \varphi, \psi\rangle = \int_{\mathbb{R}^{2n}} \big(\, f(x-y)\varphi(y)\,\big)\psi(x)\,dx\,dy.$$

Using Fubini's theorem, we get

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(z)\varphi(x-z)\psi(x)\,dx \right) dz,$$

where $z = x - y$. This gives

$$= \langle f, (R\varphi) * \psi\rangle,$$

where $(R\varphi)(z) = \varphi(z-x)$.

Definition 2.29. For $u \in \mathcal{D}'(\mathbb{R}^n)$, the convolution with a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is given by:

$$\langle u * \varphi, \psi\rangle = \langle u, (R\varphi) * \psi\rangle, \quad \forall \psi \in \mathcal{D}(\mathbb{R}^n),$$

Example 2.30.

$$\langle \delta_0 * \varphi, \psi\rangle = \langle \delta_0, (R\varphi) * \psi\rangle = \big(\,(R\varphi) * \psi\,\big)(0) = \int_{\mathbb{R}^n} \varphi(y)\psi(y)\,dy = \langle \varphi, \psi\rangle.$$

Hence, $\delta_0 * \varphi = \varphi$ in $\mathcal{D}'(\mathbb{R}^n)$.

Theorem 2.31. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $f, g \in L^1_{loc}(\Omega)$. If:*

$$\int_\Omega f(x)\varphi(x)\,dx = \int_\Omega g(x)\varphi(x)\,dx, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

then $f = g$ a.e. Hence, functions are determined by their distributions.

Proof. (Sketch) In the case where $f, g \in C(\mathbb{R}^n)$, take a sequence $\varphi_n(x) = \varphi(y - x)$ for some $y \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} f\varphi\,dx = (f * \varphi_n)(y)$$

and satisfies the desired equality

$$\int_{\mathbb{R}^n} |\varphi_n|\,dx = 1, \quad \forall n \in \mathbb{N}, \quad \varphi_n \to \delta_0 \text{ as } n \to \infty \text{ in } \mathcal{D}'(\mathbb{R}^n),$$

i.e. φ_n is a sequence of mollifiers (Homework). Then

$$\int_{\mathbb{R}^n} f\varphi_n\,dx = \int_{\mathbb{R}^n} g\varphi_n\,dx, \quad \forall \varphi_n \in \mathcal{D}'(\mathbb{R}^n).$$

$$= f * \varphi_n = g * \varphi_n.$$

As $n \to \infty$,

$$f = g = g * \delta_0 \quad \text{on compact} \quad \Omega \subset \mathbb{R}^n.$$

More generally, one needs to show that

$$(f * \varphi_n) \to f \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ uniformly.}$$

□

2.3. **Fundamental Solutions.** Let $L = \sum_{|\alpha| \le m} a_\alpha(x)\nabla_x^\alpha$ be a linear differential operator with $a_\alpha \in C^\infty(\Omega)$, $\forall \alpha \in \mathbb{N}_0$. Then its adjoint is:

$$L^*\varphi = \sum_{|\alpha| \le m} (-1)^{|\alpha|}\nabla_x^\alpha(a_\alpha\varphi).$$

Definition 2.32. Consider the PDE $Lu = f$ on Ω with $f \in \mathcal{D}'(\Omega)$.

(1) A function $u \in C^m(\Omega)$ solving $Lu = f$ pointwise for all $x \in \Omega$ is called a classical solution of the PDE.

(2) A $u \in \mathcal{D}'(\Omega)$ which satisfies $Lu = f$ in $\mathcal{D}'(\Omega)$, i.e.,

$$\langle u, L^*(\varphi)\rangle = \langle f, \varphi\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

is called a **distributional solution**.

(3) A distributional solution $u \in L^1_{\text{loc}}(\Omega)$ is called a **weak solution**.

Lemma 2.33. *Every classical solution $u \in C^m(\Omega)$ to $Lu = f$ is also a classical solution. Conversely, if $u \in \mathcal{D}'(\Omega)$ is a distributional solution and $u \in C^m(\Omega)$, then u is a classical solution.*

Proof. Homework □

Definition 2.34. Let L be as before and $x_0 \in \Omega$. A $u_{x_0} \in \mathcal{D}'(\Omega)$ satisfying $Lu_{x_0} = \delta_{x_0}$ is called a **fundamental solution** of L with singularity at x_0.

Example 2.35. Let $L = \frac{d^2}{dx^2}$ on $\mathbb{R}$. Then $u_0(x) = \frac{1}{2}|x|$ is a fundamental solution with singularity at $x_0 = 0$.

$$\langle Lu_0, \varphi\rangle = \frac{1}{2}\langle |\cdot|'', \varphi\rangle = \frac{(-1)^n}{2}\langle |\cdot|, \varphi''\rangle = \frac{1}{2}\int_{\mathbb{R}} |x|\varphi''(x)\, dx$$

Integrating by parts twice, we find

$$\langle Lu_0, \varphi\rangle = \varphi(0) = \delta_0(\varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Remark 2.36. Note that $u(x) = \frac{1}{2}|x| + kx + k$, $k \in \mathbb{R}$, is also a fundamental solution of $L = \frac{d^2}{dx^2}$. This implies there is no uniqueness.

Lemma 2.37. *If L is a linear differential operator with constant coefficients and $u_0 \in \mathcal{D}'(\Omega)$ is a fundamental solution with singularity at $x_0 = 0$, then $\tau_{x_0}u_0 \in \mathcal{D}'(\Omega)$, is a fundamental solution with singularity at x_0 where τ_{x_0} denotes translation by x_0.*

Proof.

$$\langle L(\tau_{x_0})u_0, \varphi\rangle = \langle \tau_{x_0}u_0, L^*(\varphi)\rangle = \langle u, L^*(\varphi)\circ \tau_{x_0}^{-1}\rangle = \langle u, L^*(\varphi\circ\tau_{-x_0})\rangle = \langle L(u), \varphi\circ\tau_{-x_0}$$
$$= \langle \delta_o, \varphi\circ\tau_{x_0}\rangle = \delta_{x_0}(\varphi).$$

□

Remark 2.38. Sometimes $\tau_{x_0}u$ is indeed a function $u(\cdot, x_0) \in L^1_{\text{loc}}(\Omega)$, such that

$$\varphi(x_0) = \langle u(\cdot, x_0), L^*(\varphi)\rangle = \int_\Omega u(x - x_0)L^*(\varphi)(x)\, dx.$$

Such a $u(\cdot, x_0) \in L^1_{\text{loc}}$ is called a Green's function of L.

Theorem 2.39. *Let L be a differential operator with constant coefficients and $f \in \mathcal{D}(\mathbb{R})$. If $u_0 \in \mathcal{D}'(\mathbb{R}^n)$ solves $Lu_0 = \delta_0$, then $u = u_0 * f \in \mathcal{D}'(\mathbb{R}^n)$ solves $Lu = f$ in $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. By the property of convolution,

$$\langle L(u_0 * f), \varphi\rangle = \langle L(u_0) * f, \varphi\rangle = \langle \delta_0 * f, \varphi\rangle$$

By 2.30,

$$= \langle f, \varphi\rangle.$$

□

Example 2.40. Let $f \in \mathcal{D}(\mathbb{R})$ and consider $u'' = f$ on $\mathbb{R}$. Then

$$u(x) = \frac{1}{2}(|\cdot| * f)(x) = \frac{1}{2}\int_{\mathbb{R}} |x - y| f(y)\, dy$$

solves the equation. Moreover, since $f \in \mathcal{D}(\mathbb{R})$, we have $u \in C^\infty(\mathbb{R})$ and thus a classical solution.

Remark 2.41. If $f \in L^1(\mathbb{R}^n)$ with compact support, we obtain $u \in C(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$ and thus a weak solution.

Theorem 2.42 (Malgrange–Ehrenpreis)**.** *Any linear differential operator L with constant coefficients admits a fundamental solution.*

Remark 2.43. Consider the IVP:

$$\begin{cases} \partial_t u = Lu, & t > 0 \\ u(0, x) = g \in \mathcal{D}(\mathbb{R}^n) \end{cases}$$

A fundamental solution with singularity at 0 satisfies:

$$\partial_t u - Lu = \delta_{t=0}\delta_{x=0}.$$

However, we usually find a function $F = F(t,x)$, $t > 0, x \in \mathbb{R}^n$, sufficiently differentiable such that:

$$\partial_t F = LF, \quad \text{for } t > 0, x \in \mathbb{R}^n$$
$$\lim_{t\to 0^+} F(t,x) = \delta_{x=0} \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

Then, the solution $u(t,x)$ is given by:

$$u(t,x) = \big(F(t,\cdot) * g \big)(x) = \int_{\mathbb{R}^n} F(t, x-y) g(y)\, dy.$$

Formally,

$$\partial_t u = \partial_t \big(F(t,\cdot) * g \big) = (LF) * g = L(F * g) = Lu, \quad t > 0$$
$$\lim_{t\to 0^+} u(t,\cdot) = \lim_{t\to 0^+} \big(F(t,\cdot) * g \big) = \delta_0 * g = g$$

Note: F is sometimes called a fundamental solution of the IVP.

3. Fourier Transform

3.1. The Schwartz Space and the Fourier Transform.

Definition 3.1. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the vector space of rapidly decreasing smooth functions

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) \mid \sup_{x\in\mathbb{R}^n} |x^\alpha \nabla_x^\beta f(x)| < \infty, \quad \forall \alpha, \beta \in \mathbb{N}^n \right\}.$$

Note: $C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$.

Definition 3.2. The topology on $\mathcal{S}(\mathbb{R}^n)$ is defined using the semi-norms:

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \nabla_x^\beta f(x)| \text{ for } \alpha, \beta \in \mathbb{N}^n.$$

i.e. For a sequence $\{\varphi_k\}$ in $\mathcal{S}(\mathbb{R}^n)$, we say $\varphi_k \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$ if:

$$\|\varphi_k - \varphi\|_{\alpha,\beta} \to 0, \quad \text{as } k \to \infty, \quad \forall \alpha, \beta \in \mathbb{N}^n$$

Proposition 3.3. *Let $f \in \mathcal{S}(\mathbb{R}^n)$:*

(1) $\forall N > 0, \forall \alpha \in \mathbb{N}^n, \exists C_{\alpha,N} > 0$ *such that*

$$|\nabla_x^\alpha f(x)| \le C_{\alpha,N}(1+|x|)^{-N}, \quad \forall x \in \mathbb{R}^n$$

(2) $\mathcal{S}(\mathbb{R}^n)$ *is closed under multiplication by polynomials. If $f \in \mathcal{S}, p$ is a polynomial, then $pf \in \mathcal{S}$.*

(3) $\mathcal{S}(\mathbb{R}^n)$ *is closed under differentiation.*

(4) $\mathcal{S}(\mathbb{R}^n)$ *is an algebra: if $f, g \in \mathcal{S}$, then $fg \in \mathcal{S}$.*

(5) $C_0^\infty(\mathbb{R}^n)$ *is a dense subspace of $\mathcal{S}(\mathbb{R}^n)$.*

(6) $\mathcal{S}(\mathbb{R}^n)$ *is a dense subspace of $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$.*

For (6), notice

$$C_0^\infty \subset \mathcal{S} \subset L^p$$

where the inclusion $C_0^\infty \subset L^p$ is dense.

Proof. (Proof of (5)) Let $f \in \mathcal{S}(\mathbb{R}^n)$. Consider $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ for $|x| \le 1$, and define:

$$f_k(x) = \varphi\left(\frac{x}{k}\right) f(x), \quad k \ge 1$$

Then $f_k \in C_0^\infty(\mathbb{R}^n)$ and

$$f_k(x) - f(x) = \left(\varphi\left(\frac{x}{k}\right) - 1\right) f(x)$$

which is zero for $|x| \le k$.

Since $|f(x)| \le C_N(1+|x|)^{-N}$, $N \ge 0$, for $|x| > k$:

$$|f_k(x) - f(x)| \le C_N \left(\sup_{x\in\mathbb{R}^n} |\varphi(x)| + 1\right)(1+|x|)^{-N}$$

For $|x| \le k$,

$$|f_k(x) - f(x)| = 0 \le C_N \left(\sup_{x\in\mathbb{R}^n} |\varphi(x)| + 1\right)(1+|x|)^{-N}.$$

Taking $k \to \infty$, we get $\sup_{x\in\mathbb{R}^n} |f_k - f| \to 0$, and similarly for all other semi-norms. □

Example 3.4. The function $e^{-|x|^2}$ is in $\mathcal{S}(\mathbb{R}^n)$ but not in $\mathcal{D}(\mathbb{R}^n)$.

Definition 3.5 (Fourier Transform)**.** For $f \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx$$

and we write $\hat{f} = \mathcal{F}f$.

We directly see

$$|\hat{f}(\xi)| = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(x)|\, dx < \infty \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n).$$

Lemma 3.6. *For $f \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f}$ is a bounded function on $\mathbb{R}^n$ such that*

$$\|\hat{f}\|_\infty \leq \frac{1}{(2\pi)^{n/2}} \|f\|_{L^1}.$$

Remark 3.7. Indeed, the Riemann-Lebesgue lemma shows that

$$\hat{f}(\xi) \to 0 \quad \text{as } |\xi| \to \infty.$$

Proposition 3.8. *For $f \in \mathcal{S}(\mathbb{R}^n)$, we have $\hat{f} \in C^\infty(\mathbb{R}^n)$ and the following properties hold:*

(1) *Translation:* $\tau_{-y} f(x) = f(x+y) \xrightarrow{\mathcal{F}} e^{iy\cdot\xi} \hat{f}(\xi)$ *for* $y \in \mathbb{R}^n$.

(2) *Modulation:* $e^{iy\cdot x} f(x) \xrightarrow{\mathcal{F}} \hat{f}(\xi - y) = \tau_y \hat{f}(\xi)$.

(3) *Scaling:* $f(Ax) \xrightarrow{\mathcal{F}} \frac{1}{|\det A|} \hat{f}(A^{-1}\xi)$ *for* $A \in \mathbb{R}^{n\times n}$ *invertible.*

(4) *Differentiation:* $\nabla_x^\alpha f(x) \xrightarrow{\mathcal{F}} (i\xi)^\alpha \hat{f}(\xi)$ *for all* $k \in \mathbb{N}_0$.

(5) $x^\alpha f(x) \mapsto (i\nabla_\xi)^\alpha \hat{f}(\xi)$ *for all* $\alpha \in \mathbb{N}_0$

Example 3.9. [Fourier Transform of Gaussians] Let $u = 1$ and note that $u(x) = e^{-\frac{x^2}{2}}$ satisfies $u' = -xu$. Taking the Fourier transform, we get

$$\xi \hat{u} = -\hat{u}'$$

$$\hat{u}(\xi) = ce^{-\xi^2},$$

where $c = \hat{u}(0)$. Since

$$\hat{u}(0) = c = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx,$$

we have

$$\begin{aligned} c^2 &= \frac{1}{2\pi} \int\int_{\mathbb{R}^2} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\, dxdy \\ &= \int_0^\infty re^{-\frac{r^2}{2}}\, dr = 1, \end{aligned}$$

we obtain

$$\mathcal{F}(e^{-\frac{x^2}{2}}) = e^{-\frac{\xi^2}{2}}$$

in $n = 1$. More generally, in dimension $n \in \mathbb{N}$,

$$\mathcal{F}(e^{-\frac{|x|^2}{2}}) = e^{-|\xi|^2/2} \cdot (2\pi)^{-\frac{n}{2}}.$$

Proposition 3.10. *The Fourier transform is a continuous transform*

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

Proof.

$$
\begin{aligned}
\|\hat{f}\|_{\alpha,\beta} &= \|\xi^\alpha \nabla_\xi^\beta \hat{f}\|_{L^\infty} \\
&= \|\mathcal{F}\big(\nabla_x^\alpha (x^\beta f)\big)\|_{L^\infty} \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}} \|\nabla_x^\alpha (x^\beta f)\|_{L^1} \\
&= (2\pi)^{-\frac{n}{2}} \|(1+|x|)^{-(n+1)} (1+|x|)^{n+1} \nabla_x^\alpha (x^\beta f)\|_{L^1} \\
&\le c \|(1+|x|^{n+1}) \nabla_x^\alpha (x^\beta f)\|_{L^\infty} < \infty \\
&\le C \|f\|_{\mathcal{S}_{\tilde{\alpha},\tilde{\beta}}}.
\end{aligned}
$$

□

Lemma 3.11. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\mathcal{F}(f * g)(\xi) = (2\pi)^{n/2} \hat{f}(\xi) \hat{g}(\xi).$$

Proof.

$$(2\pi)^{n/2} \mathcal{F}(f * g) = \int \int f(x-y) g(y)\, dy\, e^{-ix\cdot\xi}\, dx$$

using Fubini's theorem,

$$= \int \left(\int f(x-y) e^{-ix\cdot\xi}\, dx \right) g(y)\ dy$$

Let $z = x - y$, then

$$
\begin{aligned}
&= \int \left(\int f(z) e^{-iz\cdot\xi}\, dz \right) g(y)\, dy \\
&= (2\pi)^{n/2} \hat{f}(\xi) \hat{g}(\xi).
\end{aligned}
$$

□

3.2. **The Fourier Transform and its Inversion.**

Theorem 3.12 (Inverse Fourier Transform)**.** *For $f \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x),$$

where

$$\mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi.$$

Proof. We want to show that

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{n/2}} \int \int \hat{f}(y) e^{-iy\xi} dy\ e^{ix\xi} d\xi = f(x).$$

To do so, let

$$k(x) = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{n/2}},$$

from 3.9,

$$\hat{k} = k \quad \text{and} \quad \int_{\mathbb{R}^n} k(x) dx = (2\pi)^{n/2} \hat{k}(0) = 1.$$

Denote

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} k(\varepsilon\xi) d\xi, \quad \text{for } \varepsilon > 0.$$

Taking the limit as $\varepsilon \to 0$, by dominated convergence theorem, we have:

$$\lim_{\epsilon\to 0} f_\varepsilon(x) = \frac{1}{(2\pi)^{n/2}} \int \hat{f}(x) e^{ix\cdot\xi} d\xi$$

Now we will show the limit equals to $f(x)$,

$$\begin{aligned} f_\varepsilon(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) e^{i(x-y)\cdot\xi} k(\varepsilon\xi) d\xi dy. \\ &= (2\pi)^{-n/2} \int f(y) \left[\int e^{-i(y-x)\cdot\xi} k(\varepsilon\xi) d\xi \right] dy. \end{aligned}$$

Since the inner integral is the Fourier transform of k_ε,

$$f_\varepsilon(x) = \varepsilon^{-n} \int f(y) k_\varepsilon \left(\frac{y-x}{\varepsilon} \right) dy.$$

Thus, as $\varepsilon \to 0$,

$$f_\varepsilon(x) = (k_\varepsilon * f)(x) \to f(x)$$

where

$$k_\varepsilon(x) = \varepsilon^{-n} k\left(\frac{x}{\varepsilon}\right) \to \delta_0 \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

□

Corollary 3.13. *The Fourier transform is a linear bijection on $\mathcal{S}(\mathbb{R}^n)$ such that*

$$\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = id.$$

Corollary 3.14. *For $f, g \in \mathcal{S}(\mathbb{R}^n)$, it holds that $(f * g) \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. By the algebra property,

$$\mathcal{F}(f * g)(\xi) = (2\pi)^{n/2} \hat{f}(\xi) \hat{g}(\xi) \in \mathcal{S}(\mathbb{R}^n).$$

Taking the inverse transform yields,

$$(f * g) \in \mathcal{S}(\mathbb{R}^n).$$

□

We wil now extend $\mathcal{F}$ to $L^p(\mathbb{R}^n)$ spaces.

Proposition 3.15. *Let X be a normed space with $E \subset X$ a dense subspace. Let Y be a Banach space. If $T : E \to Y$ is a bounded linear operator such that*

$$\|Tx\|_Y \le C\|x\|_X, \quad \forall x \in E,$$

then there exists a unique bounded extension $\widetilde{T} : X \to Y$ of T.

Proof. For uniqueness, let S be another bounded extension. If $x \in X$ and $\{x_n\}_{n\in\mathbb{N}} \subset E$ such that $x_n \to x$, then

$$\lim_{n\to\infty} (\widetilde{T} - S)x_n = (\widetilde{T} - S)x.$$

While at the same time,

$$\begin{aligned} (\widetilde{T} - S)x_n &= Tx_n - Sx_n \\ &= Tx_n - Tx_n = 0. \end{aligned}$$

So,

$$\widetilde{T}x = Sx, \quad \forall x \in X.$$

To prove existence, let $\{x_n\}_{n\in\mathbb{N}} \subset E$ and $x \in X$ as before. Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in X and T is bounded, $\{Tx_n\}_{n\in\mathbb{N}}$ is also Cauchy. By completeness of Y, there exists a limit, which we denote by $\widetilde{T}x$.

Linearity follows from the linearity of T. □

Theorem 3.16 (Riemann-Lebesgue Lemma)**.**

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

uniquely extends to a bounded linear operator

$$\mathcal{F} : L^1(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$$

such that $\forall f \in L^1(\mathbb{R}^n)$,

$$\|\hat{f}\|_\infty \leq (2\pi)^{-n/2}\|f\|_1.$$

Moreover,

$$\lim_{|\xi|\to\infty} \mathcal{F}(f)(\xi) = 0.$$

Proof. Existence follows from $\|\hat{f}\|_\infty \leq (2\pi)^{-n/2}\|f\|_1$ and the previous proposition. To show that $\hat{f}$ decays as $\xi \to \infty$, we take $g \in \mathcal{S}(\mathbb{R}^n)$ and choose $\varepsilon > 0$ such that

$$\|\hat{f} - \hat{g}\|_\infty \leq \|f - g\|_{L^1}(2\pi)^{-\frac{n}{2}} < \varepsilon.$$

Choose $R > 0$ sufficiently large to obtain

$$|g(x)| \leq \varepsilon/2 \quad \text{for} \quad |x| \geq R.$$

Thus,

$$|\hat{f}(x)| \leq \|\hat{f}(x) - g(x)\|_{L^\infty} + |g(x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for $|x| \geq R$. ∎

Lemma 3.17. *For $f, g \in \mathcal{S}$, we define*

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx.$$

Then,

$$\langle \mathcal{F}(f), g\rangle = \langle f, \mathcal{F}^{-1}g\rangle.$$

Proof.

$$\begin{aligned}\langle \mathcal{F}(f), g\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx\,\overline{g(\xi)}\, d\xi. \\ &= \int_{\mathbb{R}^{2n}} f(x)\overline{\left(\int_{\mathbb{R}} g(\xi)e^{-ix\xi}\, d\xi\right)}dx. \\ &= \langle f, \mathcal{F}^{-1}g\rangle.\end{aligned}$$

∎

Theorem 3.18 (Parseval's Formula)**.** *For $f, g \in \mathcal{S}(\mathbb{R})$,*

$$\langle \mathcal{F}(f), \mathcal{F}(g)\rangle_{L^2} \;=\; \langle f, g\rangle_{L^2}.$$

Proof. Apply the previous lemma with g replaced by $\mathcal{F}(g)$. Then

$$\langle \mathcal{F}(f), \mathcal{F}(g)\rangle_{L^2} \;=\; \langle f, \mathcal{F}^{-1}\big(\,\mathcal{F}(g)\,\big)\rangle_{L^2}.$$

Since $\mathcal{F}^{-1} \circ \mathcal{F} = id$ on $\mathcal{S}$, we get

$$\langle f, \mathcal{F}^{-1}\big(\,\mathcal{F}(g)\,\big)\rangle_{L^2} \;=\; \langle f, g\rangle_{L^2}.$$

Hence

$$\langle \mathcal{F}(f), \mathcal{F}(g)\rangle_{L^2} = \langle f, g\rangle_{L^2}.$$

∎

Theorem 3.19 (Plancherel's Theorem)**.** *Setting $g = f$ in the above result, we obtain*

$$\|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2}.$$

This shows F is an isometry on $\mathcal{S} \subset L^2$. By density, we can extend it to all of L^2. Hence, the Fourier transform defines a unitary operator on L^2.

Remark 3.20. So far, we have folliwing mappings properties

$$\mathcal{F} : L^1 \to L^\infty \quad \text{on } \mathbb{R}^n,$$
$$\mathcal{F} : L^2 \to L^2,$$

and

$$L^1 \not\leftrightarrow L^2, \quad L^2 \not\leftrightarrow L^1, \quad \mathcal{S} \subset L^1 \cap L^2.$$

Definition 3.21. If $f \in L^2$, $\hat{f}$ can be represented using an integral as the following.

- When $n = 1$,

$$\begin{aligned}\hat{f}(s) &= p.v. \int f(x)e^{-ix\xi}dx \quad \text{for a.e. } s \\ &= \lim_{N\to\infty} \int_{-N}^{N} f(x)e^{-ix\xi}dx.\end{aligned}$$

- For $n \geq 1$,

$$p.v. \int = \lim_{N\to\infty} \int_{-N}^{N} \cdots \int_{-N}^{N} e^{-ix\xi}\, dx.$$

3.3. Extension to L^p Spaces and Distributions.

Theorem 3.22 (Riesz-Thorin Interpolation)**.** *Suppose a linear operator T is bounded on two endpoints:*

$$T : L^{p_1} \to L^{r_1}, \quad T : L^{p_2} \to L^{r_2}.$$

Let $\theta \in (0, 1)$ and p, r such that $p_1 \leq p \leq p_2$, $r_1 \leq r \leq r_2$ and

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}.$$

Then T is also bounded $L^p \to L^r$, and we have the estimate

$$\|Tf\|_{L^r} \leq C_1^\theta C_2^{1-\theta} \|f\|_{L^p},$$

where C_1, C_2 are the operator norms in the two endpoint cases.

Example 3.23 (Applying to the Fourier Transform.)**.** In our setting:

- We know $\mathcal{F} : L^1 \to L^\infty$ (endpoint 1).
- We also know $\mathcal{F} : L^2 \to L^2$ (endpoint 2).

Set $(p_1, r_1) = (1, \infty)$ and $(p_2, r_2) = (2, 2)$. Then $C_1 = (2\pi)^{-\frac{n}{2}}, C_2 = 1$. For $1 < p < 2$, we define

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2} \implies \theta = \frac{2}{p} - 1.$$

Similarly,

$$\frac{1}{r} = \frac{\theta}{\infty} + \frac{1-\theta}{2} = \frac{1-\theta}{2} = 1 - \frac{1}{p}$$

Then,

$$\|Ff\|_{L^r} \leq (2\pi)^{\frac{n}{2}(1-\frac{2}{p})}\|f\|_{L^p}.$$

Proposition 3.24. *If $f \in L^p$, $1 \leq p \leq 2$, and $\hat{f} \in L^1$, then*

$$\mathcal{F}^{-1}(\hat{f}) = f.$$

Example 3.25. Let $f \in L^1(\mathbb{R})$ be defined by

$$f(x) \;=\; \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

Then its Fourier transform is

$$\hat{f}(\xi) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\,x\,\xi}\, \mathrm{d}x.$$

Because $f(x) = 1$ for $|x| \le 1$ and 0 otherwise,

$$\hat{f}(\xi) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\,x\,\xi}\, \mathrm{d}x.$$

$$\hat{f}(\xi) \;=\; \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} \cos(x\,\xi)\, \mathrm{d}x \;-\; i \int_{-1}^{1} \sin(x\,\xi)\, \mathrm{d}x \right].$$

Observe that $\sin(x\,\xi)$ is an odd function over the symmetric interval $[-1, 1]$, so the imaginary part integrates to zero. Meanwhile, the cosine part is even, giving

$$\int_{-1}^{1} \cos(x\,\xi)\, \mathrm{d}x \;=\; 2 \int_{0}^{1} \cos(x\,\xi)\, \mathrm{d}x \;=\; 2 \left[\frac{\sin(\xi)}{\xi} \right].$$

Thus,

$$\hat{f}(\xi) \;=\; \frac{1}{\sqrt{2\pi}} \cdot 2\, \frac{\sin(\xi)}{\xi} \;=\; \sqrt{\frac{2}{\pi}}\, \frac{\sin(\xi)}{\xi}.$$

For $\xi = 0$, the standard limit $\frac{\sin(\xi)}{\xi} \to 1$ applies, yielding

$$\hat{f}(0) \;=\; \sqrt{\frac{2}{\pi}}.$$

Notice that $\hat{f} \in L^2(\mathbb{R})$, but $\hat{f} \notin L^1(\mathbb{R})$, then we have $f = \mathcal{F}^{-1}(\hat{f})$ in $L^2(\mathbb{R})$. Indeed,

$$\|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2$$

yields

$$\int_{-\infty}^{\infty} \frac{\sin^2 \xi}{\xi^2} d\xi = \pi.$$

Definition 3.26. A linear continuous functional

$$u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$$

is called a **tempered distribution**, denoted as $u \in \mathcal{S}'(\mathbb{R}^n)$. We say that $u_j \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ if

$$u_j(\varphi) \to u(\varphi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Remark 3.27. Tempered distributions are distributions.

Example 3.28. Consider the following examples

(i) $\delta_{x_0} \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and there exists $N \ge 0$ such that

$$(1 + |\cdot|^2)^{-N} u \in L^1(\mathbb{R}^n),$$

then $u \in \mathcal{S}'(\mathbb{R}^n)$.

Sketch: $\varphi \in \mathcal{S}$ implies

$$\int_{\mathbb{R}^n} |u\varphi| \le \|(1 + |x|^2)^{-N} u\|_{L^1} \|(1 + |x|^2)^N \varphi\|_\infty.$$

(iii) $u(x) = e^{|x|}$. Then $u \in \mathcal{D}'(\mathbb{R}^n)$ but $u \notin \mathcal{S}'(\mathbb{R}^n)$.

Lemma 3.29. *Let $y \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$, and let $a \in C^\infty(\mathbb{R}^n)$ such that a and its derivatives grow at most polynomially. Then the operators M_a, τ_y, R, and ∂_x^α defined on $\mathcal{S}(\mathbb{R}^n)$ extend to continuous operations on $\mathcal{S}'(\mathbb{R}^n)$*

Proof. Exercise! □

Definition 3.30. For $u \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform $\mathcal{F}(u) \in \mathcal{S}'(\mathbb{R}^n)$ is defined such that

$$\langle \mathcal{F}(u), \varphi \rangle = \langle u, \mathcal{F}(\varphi) \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Remark 3.31. In addition, $F : \mathcal{S}' \to \mathcal{S}'$ is an isomorphism with inverse $\mathcal{F}^{-1}$ because

$$\langle \mathcal{F}^{-1}\mathcal{F}(u), \varphi \rangle = \langle \mathcal{F}(u), \mathcal{F}(\varphi) \rangle = \langle u, \mathcal{F}\mathcal{F}^{-1}\varphi \rangle.$$

Thus,

$$FF^{-1} = id_{\mathcal{S}'}.$$

Proposition 3.32. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$. Then*

$$\mathcal{F}(\partial_x^\alpha u) = (i\xi)^\alpha \mathcal{F}(u),$$

and

$$(x^\alpha u) = i\partial_\xi^\alpha \big(\mathcal{F}(u) \big)$$

in $\mathcal{S}'(\mathbb{R}^n)$.

Example 3.33. Let $\varphi \in \mathcal{S}$, then

$$\langle \hat{\delta}_0, \varphi \rangle = \langle \delta_0, \hat{\varphi} \rangle.$$

Since

$$\hat{\varphi}(0) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x) e^{-i0\cdot x}\, dx,$$

$$\hat{\delta}_0 = \frac{1}{(2\pi)^{n/2}} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Example 3.34. Consider the fundamental solution to the Laplacian equation:

$$-\Delta u = \delta_0 \text{ in } \mathbb{R}^n.$$

The Fourier transform of δ_0 satisfies

$$\hat{\delta}_0(\xi) = (2\pi)^{-n/2}.$$

Hence, for $\xi \neq 0$, the Fourier transform of the fundamental solution satisfies

$$|\xi|^2 \hat{u}(\xi) = (2\pi)^{-n/2},$$

$$\hat{u}(\xi) = \frac{(2\pi)^{-n/2}}{|\xi|^2}.$$

One can show that $\forall \gamma \in (0, n)$,

$$\mathcal{F}^{-1}\Big(\frac{1}{|\cdot|^\gamma}\Big) = C_{n,\gamma} \cdot \frac{1}{|x|^{n-\gamma}} \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

In particular, for $n = 3$ and $\gamma = 2$, we have

$$u(x) = \frac{1}{4\pi|x|}$$

which corresponds to the Newtonian potential.

A framework for understanding the regularity of $u \in \mathcal{S}'(\mathbb{R}^n)$ is provided by Sobolev spaces.

3.4. **Sobolev Spaces.**

Definition 3.35. The Sobolev space $H^k(\mathbb{R}^n)$ is defined as

$$H^k(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \partial^\alpha u \in L^2(\mathbb{R}^n), \forall |\alpha| \leq k\}$$

where for integer k, the derivative $\partial^\alpha u$ is understood in the sense of distributions.

Remark 3.36. $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$, where $W^{k,p}(\mathbb{R}^n)$ denotes the L^p-based Sobolev space.

Definition 3.37. An inner product on $H^k(\mathbb{R}^n)$ is given by:

$$\langle u, v\rangle_{H^k} = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^k} \partial_x^\alpha u \, \overline{\partial_x^\alpha v} \, dx.$$

Since L^2 is complete, we conclude that $H^k(\mathbb{R}^n)$ is a Hilbert space.

Lemma 3.38. *Let $P_k(\xi) = \sum_{|\alpha|\leq k} |\xi|^{2\alpha}$. Then $u \in L^2(\mathbb{R}^n)$ is in $H^k(\mathbb{R}^n)$ if and only if:*

$$\|u\|_{H^k}^2 = \int_{\mathbb{R}^n} P_k(\xi)|\hat{u}(\xi)|^2 d\xi < \infty.$$

Proof. The proof follows from 3.19 such that $\|u\|_{H^k} = \sqrt{\langle u, u\rangle_{H^k}}$, and the property that

$$F(\partial^\alpha u) = (i\xi)^\alpha \hat{u}(\xi) \text{ in } \mathcal{S}'.$$

□

Definition 3.39. Define the Japanese bracket:

$$\langle \xi \rangle = (1 + |\xi|^2)^{1/2}.$$

It satisfies for constant c_1, c_2,

$$c_1 \langle \xi \rangle^{2k} \leq P_k(\xi) \leq c_2 \langle \xi \rangle^{2k}, \quad \forall \xi \in \mathbb{R}^n.$$

Thus, we can define $P_k(\xi)$ by $\langle \xi \rangle^{2k}$ to obtain an equivalent norm.

Definition 3.40. For $s \in \mathbb{R}$, we define:

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{H^s} < \infty\}$$

where:

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

This defines a *weighted* L^2-norm. Then, the inner product is given by:

$$\langle u, v\rangle_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)} d\xi.$$

Remark 3.41. We have the following properties.

- The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.
- If $s > r$, then $H^s(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$.
- For $s > 0$, we have $(1 + |\xi|^2)^s \geq 1$, so $u \in H^s(\mathbb{R}^n)$ implies $u \in L^2(\mathbb{R}^n)$. Since

  $$\|u\|_{L^2}^2 = \|\hat{u}\|_{L^2}^2 = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

  For $s < 0$, the elements of $H^s(\mathbb{R}^n)$ are distributions rather than functions.

Example 3.42. The Dirac delta δ_0 is in $H^s(\mathbb{R}^n)$ for all $s < -\frac{n}{2}$, since

$$\hat{\delta}_0 = (2\pi)^{-\frac{n}{2}},$$

we have

$$\|\delta_0\|_{H^s}^2 = (2\pi)^n \int_{\mathbb{R}^n} (1+|\xi|^2)^s d\xi.$$

Using polar coordinates yields

$$\|\delta_0\|_{H^s}^2 = c_{n,s} \int_0^\infty (1+r^2)^s r^{n-1} dr < \infty$$

if and only if $n-1+2s < -1$, which simplifies to $s < -\frac{n}{2}$.

Definition 3.43. Define the space

$$C_{00}^k(\mathbb{R}^n) = \{u \in C^k(\mathbb{R}^n) : \lim_{|x|\to\infty} |\partial^\alpha u(x)| = 0, \forall |\alpha| \le k\}.$$

with the norm

$$\|u\|_{C_{00}^k} = \max_{|\alpha|\le k} \|\partial^\alpha u\|_{L^\infty}.$$

Remark 3.44. $C_{00}^k(\mathbb{R}^n)$ is a closed subspace of $C^k(\mathbb{R}^n)$, implying it is a Banach space.

Remark 3.45. By the Remark (Riemann-Lebesgue Lemma, if $\hat{f} \in L^1(\mathbb{R}^n)$, then $f \in C_{00}(\mathbb{R}^n)$.

Theorem 3.46 (Sobolev Embedding Theorem)**.** *For $sp < n$, we have*

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \quad \text{where} \quad q = \frac{np}{n-sp}.$$

The embedding holds for all finite q but fails at the endpoint $q = \infty$, $ps = n$.

For $ps > n$, let $\alpha = s - \frac{n}{p}$, then

$$W^{s,p} \hookrightarrow C^\alpha,$$

where

$$C^\alpha = \{u \in \mathcal{S}' : u \in L^\infty, \ |u(x) - u(y)| \le c|x-y|^\alpha\}.$$

Theorem 3.47. *Let $s > \frac{n}{2}$. Then we have the continuous embedding*

$$H^s(\mathbb{R}^n) \ \hookrightarrow \ C^k(\mathbb{R}^n) \quad \text{for any integer } k < s - \frac{n}{2}.$$

Proof. If $u \in H^s(\mathbb{R}^n)$, then we wish to show $\partial^k u \in L^\infty(\mathbb{R}^n)$ (hence $u \in C^k$). Observe that

$$\widehat{\partial^k u}(\xi) \ = \ (i\,\xi)^k\, \widehat{u}(\xi),$$

so by the inverse Fourier transform,

$$\partial^k u(x) \ = \ \int_{\mathbb{R}^n} (i\xi)^k\, e^{i\,x\cdot\xi}\, \widehat{u}(\xi)\, d\xi.$$

Taking absolute values,

$$|\partial^k u(x)| \ \le \ \int_{\mathbb{R}^n} |\xi|^k\, |\widehat{u}(\xi)|\, d\xi.$$

$$= \int_{\mathbb{R}^n} \frac{\langle\xi\rangle^k}{\langle\xi\rangle^s}\, \langle\xi\rangle^s\, |\widehat{u}(\xi)|\, d\xi.$$

Apply the Cauchy–Schwarz inequality:

$$|\partial^k u(x)| \ \le \ \left(\int_{\mathbb{R}^n} \frac{d\xi}{\langle\xi\rangle^{2(s-k)}}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s}\, |\widehat{u}(\xi)|^2\, d\xi\right)^{\frac{1}{2}}.$$

Because $s - k > \frac{n}{2}$, the first integral converges, while the second integral is just the H^s-norm of u. Thus

$$\|\partial^k u\|_{L^\infty(\mathbb{R}^n)} \ \le \ C\, \|u\|_{H^s},$$

implying $\partial^k u$ is bounded (and continuous via standard Fourier inversion arguments). Therefore $u \in C^k(\mathbb{R}^n)$ whenever $k < s - \frac{n}{2}$. $\square$

Lemma 3.48. *If $s > \frac{n}{2}$, then $H^s(\mathbb{R}^n)$ is a Banach algebra; i.e.,*

$$\| u\, v\|_{H^s} \ \leq\ C_s \, \|u\|_{H^s} \, \|v\|_{H^s} \quad \textit{for all } u, v \in H^s(\mathbb{R}^n).$$

Proof. When $s > \frac{n}{2}$, we already know $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ and that $\|\partial^k u\|_{L^\infty(\mathbb{R}^n)} \leq C\, \|u\|_{H^s},$. Then for multi-indices α, we have

$$\partial^\alpha(uv) \ =\ \sum_{\beta+\gamma=\alpha} \partial^\beta u \, \partial^\gamma v,$$

and

$$\begin{aligned} \|\partial^\alpha(u\, v)\|_{L^2} \ &\leq\ \sum_{\beta+\gamma=\alpha} \|\partial^\beta u \ \partial^\gamma v\|_{L^2}. \\ &\leq C\|u\|_{H^s}\|v\|_{H^s} \end{aligned}$$

Summing over all multi-indices with $|\alpha| \leq s$ proves the result. $\square$

4. The Heat Equation

4.1. **Background.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and define $\rho(t,x) > 0$ as the energy density at time t and location $x \in \Omega$. The total energy in Ω at time t is

$$E(t) \;=\; \int_\Omega \rho(t,x)\,dx.$$

Its rate of change is

$$\dot{E}(t) \;=\; \int_\Omega \partial_t \rho(t,x)\,dx.$$

According to the principle of conservation of energy, this must equal the net heat flux through the boundary $\partial\Omega$. If $J(t,x)$ denotes the heat flux vector, then

$$\dot{E}(t) \;=\; -\int_{\partial\Omega} J(t,x)\cdot \vec{\nu}\,ds,$$

where $\vec{\nu}$ is the outward unit normal on $\partial\Omega$.

By the divergence theorem,

$$\int_{\partial\Omega} J(t,x)\cdot n\,ds \;=\; \int_\Omega \nabla\cdot J(t,x)\,dx.$$

Hence,

$$\dot{E}(t) \;=\; \int_\Omega \partial_t\rho(t,x)\,dx \;=\; -\int_\Omega \nabla\cdot J(t,x)\,dx.$$

From this, we deduce the local form of energy conservation:

$$\partial_t\rho \;+\; \nabla\cdot J \;=\; 0.$$

Fourier's law states that the flux is proportional to the negative of the temperature (or energy) gradient

$$J \;=\; -\,k\,\nabla\rho, \quad k > 0.$$

Substituting this into the continuity equation $\partial_t\rho + \nabla\cdot J = 0$ yields

$$\partial_t\rho \;-\; k\,\Delta\rho \;=\; 0,$$

the classical **heat equation**.

A more general constitutive relation sometimes used is

$$J \;=\; -\,k\,\nabla(\rho^m), \quad m \geq 0.$$

In that case,

$$\partial_t\rho \;=\; M\,k\,\nabla\cdot\big(\rho^{\,m-1}\,\nabla\rho\big).$$

- The quantity $m\,\rho^{m-1}$ acts as a diffusivity term.
- When $m = 1$, we recover the standard heat equation with constant diffusivity.
- When $m < 1$, the factor ρ^{m-1} tends to zero as $\rho \to 0$, leading to slow diffusion.
- When $m > 1$, this corresponds to fast diffusion.

Definition 4.1. Let $I \subset \mathbb{R}$ be an open interval. A function

$$u : I \;\to\; \mathcal{S}(\mathbb{R}^n)$$

is continuous on I if, for every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$,

$$\|u(t+h) \;-\; u(t)\|_{\alpha,\beta} \;\xrightarrow[h\to 0]{}\; 0.$$

Definition 4.2. We denote the set of such functions by

$$C\big(I;\mathcal{S}(\mathbb{R}^n)\big).$$

We say $u \in C^1\big(I;\mathcal{S}(\mathbb{R}^n)\big)$ if $u, \dot{u} \in C\big(I;\mathcal{S}(\mathbb{R}^n)\big)$ where $\dot{u}$ is such that

$$\left\|\frac{1}{h}\big(u(t+h) \;-\; u(t)\big) \;-\; \dot{u}(t)\right\|_{\alpha,\beta} \xrightarrow[h\to 0]{} 0, \quad \forall \alpha,\beta \in \mathbb{N}^n.$$

We can then define $C^k\big(I;\mathcal{S}(\mathbb{R}^n)\big)$ recursively for all $k \geq 0$.

Remark 4.3. A function $u \in C\big(I;\mathcal{S}(\mathbb{R}^n)\big)$ defines a function

$$v : I \times \mathbb{R}^n \;\to\; \mathbb{R}$$

such that

$$v(t,x) \;=\; u(t)(x).$$

Then $v \in C\big(I \times \mathbb{R}^n\big)$, and $v(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$. One identifies $u(t)(x)$ with $v(t,x)$ and so we write $\partial_t u(t,\cdot)$ instead of $\dot{u}(t)$.

Lemma 4.4. *Let $u \in C^k\big(I;\mathcal{S}(\mathbb{R}^n)\big)$. Define*

$$\hat{u} \;=\; \mathcal{F}u : I \;\to\; \mathcal{S}(\mathbb{R}^n)$$

by

$$\hat{u}(t) \;=\; \mathcal{F}\big(u(t)\big) = \widehat{u(t)}.$$

Then $\hat{u}$ also lies in $C^k\big(I;\mathcal{S}(\mathbb{R}^n)\big)$, and for $0 \leq j \leq k$,

$$\partial_t^j \hat{u}(t) \;=\; \mathcal{F}\Big(\partial_t^j u(t)\Big).$$

Proof. Exercise. □

4.2. **Heat Equation Solution by Fourier Transform.** Consider the initial value problem (IVP) for the heat equation in $\mathbb{R}^n$:

$$\begin{cases} \partial_t u(t,x) \;-\; \Delta u(t,x) \;=\; f(t,x), & t > 0, \\ u(0,x) \;=\; u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

where $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and $f \in C^\infty\big(\mathbb{R};\mathcal{S}(\mathbb{R}^n)\big)$.

Remark 4.5. The operator $L \;=\; \partial_t \;-\; \Delta$ is second-order but we only have 1 initial data, so this is not a Cauchy Problem.

Remark 4.6. $\Gamma \;=\; \{(t,x) \in \mathbb{R}^{1+n} \;:\; t \;=\; 0\}$ is characteristic because for the principal symbol

$$L_p(\xi,\xi_1,\ldots,\xi_n) \;= -\sum_{j=1}^{n} \xi_j^2$$

and $\vec{v} = (1,0,\ldots,0)$ satisfies $L_p(\vec{v}) = 0$.

Example 4.7.

$$u(t,x) \;=\; \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}\,\frac{d^k}{dt^k}\Big(e^{-\frac{1}{t^2}}\Big) \not\equiv 0$$

solves

$$\begin{cases} \partial_t u \;-\; \Delta u \;=\; 0, \\ u(0,x) \;=\; 0, \end{cases}$$

which yields non-unique solutions.

Theorem 4.8. *There exists a unique classical solution*

$$u \in C^\infty\big([0,\infty);\mathcal{S}(\mathbb{R}^n)\big).$$

Moreover, the solution map

$$s(t): u_0 \mapsto u(t,\cdot)$$

is continuous. Notice this implies the well-posedness of the solution.

Proof. Using the Fourier transform in $x \in \mathbb{R}^n$ yields

$$\begin{cases} \partial_t \hat{u} + |\xi|^2 \hat{u} = \hat{f}(t,\xi), & t > 0 \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), & \hat{u}_0 \in \mathcal{S} \end{cases}$$

Recall that for $u(t) \in \mathcal{S}$ we have $\hat{u}(t) = \widehat{u(t)}$ and $\partial_t \hat{u} = \widehat{\partial_t u}$. Thus, we can solve the t-dependent ODE with ξ as a parameter. There exist the unique solution

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{u}_0(\xi) + \int_0^t e^{-(t-s)|\xi|^2}\hat{f}(s,\xi)\,ds, \quad t > 0, \xi \in \mathbb{R}^n.$$

Since $\hat{u} \in C^\infty([0,\infty);\mathcal{S}(\mathbb{R}^n))$, we have $u = \mathcal{F}^{-1}\hat{u}$ solves 4.1. ☐

Corollary 4.9 (Duhamel's Formula)**.** *For $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and $f \in C^\infty\big([0,\infty),\mathcal{S}(\mathbb{R}^n)\big)$, the unique classical solution*

$$u \in C^\infty\big([0,\infty);\mathcal{S}(\mathbb{R}^n)\big)$$

is given by

$$u(t,x) = (K_t * u_0)(x) + \int_0^t \big(K_{t-s} * f(s,\cdot)\big)(x)\,ds, \tag{4.2}$$

where the heat kernel is

$$K_t(x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, \quad t > 0.$$

Proof. For $t > 0$, the Fourier transform of the heat kernel satisfies

$$\mathcal{F}^{-1}(e^{-t|\xi|^2}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad \forall x \in \mathbb{R}^n.$$

Thus,

$$(2\pi)^{n/2} K_t(x) = e^{-\frac{|x|^2}{4t}}.$$

☐

Remark 4.10. More generally, 4.2 extends to the case

$$f, \partial_t f, \partial_x^\alpha f, |\alpha| \le 2: \text{ constant and bounded}$$

with classical solution

$$u \in C^{1,2}\big([0,\infty)\times\mathbb{R}^n\big) \cap \big(C([0,\infty)\times\mathbb{R}^n)\big).$$

Proposition 4.11 (Properties of the Heat Kernel)**.** *We have the following properties.*

(a) $t \mapsto K_t \in C^\infty\big((0,\infty);\mathcal{S}(\mathbb{R}^n)\big)$.

(b) $\partial_t K_t - \Delta K_t = 0$ *on* $(0,\infty)\times\mathbb{R}^n$.

(c) $\|K_t\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} K_t(x)\,dx = 1 = (2\pi)^{n/2}\mathcal{F}(K_t)(0)$.

(d) *Semigroup property:* $K_s * K_t = K_{s+t}, \quad \forall s,t > 0$.

(e) $K_t \to \delta_0$ *in* $\mathcal{D}'(\mathbb{R}^n)$ *as* $t \to 0^+$.

Proof. Sketch of (e): For $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}&\langle K_t, \varphi\rangle - \langle \delta_0, \varphi\rangle \\ &= \int_{\mathbb{R}^n} K_t(x)\varphi(x)\,dx - \varphi(0) \\ &= \int_{\mathbb{R}^n} K_t(x)\big(\varphi(x) - \varphi(0)\big)\,dx.\end{aligned}$$

□

Theorem 4.12. *K_t yields the fundamental solution to the heat equation's initial value problem, in the sense that*

$$\widetilde{K}_t(x) = \begin{cases} K_t(x), & t > 0 \\ 0, & t = 0 \end{cases}$$

solves

$$(\partial_t - \Delta)\widetilde{K}_t = \delta_{t=0}\delta_{x=0}.$$

Proof. For $t > 0$, $x \neq 0$, clearly we have

$$(\partial_t - \Delta)\widetilde{K}_t = (\partial_t - \Delta)K_t = 0,$$

by (b) from the previous proposition. Hence all we need to show is

$$\langle \widetilde{K}_t, L^*(\varphi)\rangle = \langle \widetilde{K}_t, -(\partial_t + \Delta)\varphi\rangle = \varphi(0,0), \quad \forall \varphi \in D(\mathbb{R}^{n+1}).$$

To do so, let $\varepsilon \geq 0$ then

$$\langle \widetilde{K}_t, L^*(\varphi)\rangle = -\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \int_{\mathbb{R}^n} K_t(x)(\partial_t + \Delta)\varphi(t,x)\,dx\,dt$$

Using integration by parts

$$\begin{aligned}\langle \widetilde{K}_t, L^*(\varphi)\rangle = &-\lim_{\varepsilon\to 0}\left[\int_{\mathbb{R}^n}\int_\varepsilon^\infty (\partial_t - \Delta)K_t(x)\varphi(t,x)\,dt\,dx\right] \\ &+ \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} K_\varepsilon(x)\varphi(\varepsilon,x)\,dx.\end{aligned}$$

Since $(\partial_t - \Delta)K_t(x) = 0$, and $K_t \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $t \to 0^+$ by the previous proposition, we have

$$\langle \widetilde{K}_t, L^*(\varphi)\rangle = \varphi(0,0).$$

□

4.3. **Further Properties.**

Theorem 4.13. *Let $u \in C^\infty(\mathbb{R}; S(\mathbb{R}^n))$ be the unique solution to 4.1 with $f \equiv 0$. Denote*

$$E(t) = \int_{\mathbb{R}^n} u(t,x)\,dx,$$

then

$$E(t) = E(0) \quad \forall t \geq 0.$$

Proof. Notice that

$$E(t) = \int_{\mathbb{R}^n} u(t,x)\,dx = (2\pi)^{\frac{n}{2}}\hat{u}(t,0).$$

Since $\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{u}_0(\xi) + \int_0^t e^{-(t-s)|\xi|^2}\hat{f}(s,\xi)\,ds$ and $f \equiv 0$, letting $\xi = 0$ yields

$$\begin{aligned}E(t) &= (2\pi)^{\frac{n}{2}}\hat{u}_0(\xi)e^{-t|\xi|^2}\Big|_{\xi=0} \\ &= (2\pi)^{\frac{n}{2}}\hat{u}_0(0)\end{aligned}$$

$$= \int_{\mathbb{R}^n} u_0(x)\,dx$$
$$= \int_{\mathbb{R}^n} u(0,x)\,dx = E(0)$$

□

Remark 4.14 (Young's Inequality)**.** Let $p, q, r \in [1, \infty)$ s.t. $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then if $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$,

$$(f * g) \in L^r(\mathbb{R}^n)$$

and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

The special case $p = 1, r = q$ follows from Minkowski's inequality for integrals.

The case $p = \frac{q}{q-1}, r = \infty$ follows from Hölder's inequality The rest follows from interpolation.

Proposition 4.15. *Let $u \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$ be the solution to 4.1 with $f \equiv 0$. Then*

$$\lim_{t \to +\infty} \|u(t, \cdot)\|_{L^p} = 0, \quad \forall p > 1.$$

Proof. Since $\forall t \geq 1$,

$$K_t \in S(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n),$$

and

$$\|K_t\|_{L^1} \lesssim |t|^{-\frac{n}{2}(1-\frac{1}{p})}. \quad \text{(Homework)}$$

We obtain, using Young's inequality,

$$\|u(t, \cdot)\|_{L^p} = \|K_t * u_0\|_{L^p}$$
$$\leq \|K_t\|_{L^1} \|u_0\|_{L^p},$$

where $\|u_0\|_{L^p} < \infty$ since $u_0 \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\|u(t, \cdot)\|_{L^p} \leq |t|^{-\frac{n}{2}(1-\frac{1}{p})} \|u_0\|_{L^p} \xrightarrow[t \to \infty]{} 0.$$

□

Proposition 4.16. *The propagator*

$$S(t) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n),$$
$$u_0 \mapsto K_t * u_0,$$

uniquely extends to a map $S(t) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Proof. Young's inequality shows that for $u_0 \in L^p$, $\forall t \geq 0$,

$$(K_t * u_0) \in L^p(\mathbb{R}^n),$$

yielding the extension of $S(t)$ on $L^p(\mathbb{R}^n)$. Uniqueness follows from $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, and if two linear, bounded (continuous) operators on a Banach space coincide on a dense subspace, then they coincide everywhere. □

Corollary 4.17. *Let $u_0 \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, then*

$$S(t)u_0 \in C^\infty\big((0, \infty) \times \mathbb{R}^n\big)$$

is a smooth solution to the heat equation with $f \equiv 0$.

Proof. Follows from the fact that $K_t \in \mathcal{S}(\mathbb{R}^n)$ for all $t > 0$, and

$$S(t)u_0 = K_t * u_0.$$

□

Remark 4.18. This shows the strong regularization property of the heat equation.

Proposition 4.19 (Infinite speed of propogation)**.** *Let* $f \equiv 0, u_0 \in \mathcal{S}(\mathbb{R}^n)$*,* $u_0 \geq 0$*. Then for all* $t > 0$*, the solution* $u(t,x) > 0$ *unless* $u_0 \equiv 0$*.*

Proof. Notice that

$$u(t,x) = \underbrace{\frac{1}{(4\pi t)^{n/2}}}_{>0, \forall t>0} \int_{\mathbb{R}^n} \underbrace{e^{-\frac{|x-y|^2}{4t}}}_{>0 \text{ in} x} \underbrace{u_0(y)}_{\geq 0} \, dy > 0$$

unless $u_0 \equiv 0$. □

5. The Wave Equation

5.1. Background and Basics.

Definition 5.1. For $(t,x) \in \mathbb{R}^{1+n}$, the wave equation is

$$\begin{cases} \partial_{tt} u - c^2 \Delta u = f(t,x) \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \end{cases} \tag{5.1}$$

where $c > 0$ is the local wave speed.

Motivation: Maxwell's Equations of Electrodynamics in absence of charges

$$\nabla \cdot E = 0, \quad \nabla \times E = -\partial_t B$$

$$\nabla \cdot B = 0, \quad \nabla \times B = \mu_0 \varepsilon_0 \partial_t E$$

where μ_0, ε_0 are the electromagnetic permeability and permittivity. Formally,

$$\begin{aligned} \nabla \times (\nabla \times E) &= -\partial_t (\nabla \times B) \\ &= -\mu_0 \varepsilon_0 \partial_{tt} E. \end{aligned}$$

Moreover, using the identity

$$\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \Delta E$$

we obtain:

$$\partial_{tt} E = c^2 \Delta E, \quad \text{for } c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}},$$

and similarly for B.

For simplicity, we set $c = 1$ (i.e., rescale $t \mapsto ct$) and study the non-characteristic Cauchy problem with

$$u_0, u_1 \in \mathcal{S}(\mathbb{R}^n), \quad f \in C^\infty(\mathbb{R}; S(\mathbb{R}^n)).$$

Theorem 5.2. *Under these assumptions, there exists a unique solution*

$$u \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$$

to 5.1.

Proof. Taking the Fourier transform in x yields

$$\begin{cases} \partial_{tt} \hat{u} + |\xi|^2 \hat{u} = \hat{f}(t,\xi) \\ \hat{u}|_{t=0} = \hat{u_0}, \quad \partial_t \hat{u}|_{t=0} = \hat{u_1}. \end{cases}$$

For every $\xi \in \mathbb{R}^n$, this is a second-order ODE in t. A system of linearly independent fundamental solutions is:

$$e^{\pm it|\xi|} = \cos(t|\xi|) \pm i \sin(t|\xi|).$$

The variation of constants formula then yields

$$\hat{u}(t,\xi) = \cos(t|\xi|)\hat{u_0}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{u_1}(\xi) + \int_0^t \frac{\sin\big((t-\tau)|\xi|\big)}{|\xi|}\hat{f}(\tau,\xi)d\tau \tag{5.2}$$

the unique solution to the ODE.

Clearly, $\hat{u}$ given by 5.2 is C^∞ in time. Moreover, since $u_0, u_1 \in \mathcal{S}(\mathbb{R}^n)$, and

$$\cos(t|\xi|) = \sum_{k=0}^\infty \frac{(-1)^k (t|\xi|)^{2k}}{(2k)!},$$

this is a power series with an infinite radius of convergence and hence smooth. Similarly, for $\frac{\sin(t|\xi|)}{|\xi|}$, we conclude

$$\hat{u} \in C^\infty\big(\mathbb{R}; \mathcal{S}(\mathbb{R}^n)\big).$$

∎

Remark 5.3. 5.2 gives the solution to $\Box u = f(t,x)$ via

$$u(t,x) = \mathcal{F}^{-1}\big(\,\hat{u}(t,\xi)\,\big). \tag{5.3}$$

This formula implies the following regularity estimate, where for $\varphi \in H^s(\mathbb{R}^n)$, $\varphi \in H^{s-1}(\mathbb{R}^n)$, we denote

$$\|(\varphi,\psi)\|_{H^s\times H^{s-1}} = \|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}}.$$

Proposition 5.4. *5.2 shows that for $u_0 \in H^s, u_1 \in H^{s-1}$ and $f \in C(\mathbb{R};H^{s-1})$, the solution given by 5.3 satisfies*

$$u \in C\big(\mathbb{R};H^s(\mathbb{R}^n)\big) \cap C^1\big(\mathbb{R};H^{s-1}(\mathbb{R}^n)\big)$$

and for some $C_1, C_2 > 0$,

$$\|(u,\partial_t u)\|_{H^s\times H^{s-1}} \leq C_1\|(u_0,u_1)\|_{H^s\times H^{s-1}} + C_2\int_0^t \|f(\tau,\cdot)\|_{H^{s-1}}d\tau.$$

Remark 5.5. This shows that for $u_0 \in H^s$, we have $u(t,\cdot) \in H^s$ but not better(no regularization).

Remark 5.6. By density $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$, the formula 5.2 allows us to extend the solution to a mild solution

$$u \in \underbrace{C\big(\mathbb{R};H^s(\mathbb{R}^n)\big)}_{\hookrightarrow C\left(\mathbb{R};C^s(\mathbb{R}^n)\right) \text{ for } s>\frac{n}{2}+1} \cap C^1\big(\mathbb{R};H^{s-1}(\mathbb{R}^n)\big).$$

Proof. For the proof, we recall Minkowski's inequality: for $F:\Omega_1\times\Omega_2 \to \mathbb{C}$ measurable, we have

$$\left(\int_{\Omega_1}\left|\int_{\Omega_2} F(x,y)dy\right|^p dx\right)^{\frac{1}{p}} \leq \int_{\Omega_2}\left(\int_{\Omega_1}|F(x,y)|^p dx\right)^{\frac{1}{p}} dy, \quad \text{for } p \geq 1.$$

Recall that

$$\|u\|_{H^s} = \|\langle\xi\rangle^s\hat{u}\|_{L^2},$$

for $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$, multiplying the solution formula by $\langle\xi\rangle^s$ and taking L^2-norms, we obtain using triangle inequality

$$\|\langle\xi\rangle^s\,\hat{u}(t,\xi)\|_{L^2_\xi} \;\leq\; \big\|\langle\xi\rangle^s\,\hat{u}_0(\xi)\cos(|\xi|t)\big\|_{L^2_\xi} \;+\; \big\|\langle\xi\rangle^s\,\tfrac{\hat{u}_1(\xi)}{|\xi|}\sin(|\xi|t)\big\|_{L^2_\xi}$$
$$+\left\|\langle\xi\rangle^s\int_0^t\frac{\sin((t-\tau)|\xi|)}{|\xi|}\,\hat{f}(\tau,\xi)\,d\tau\right\|_{L^2_\xi}.$$

Notice that

$$\big\|\langle\xi\rangle^s\,\hat{u}_0(\xi)\cos(|\xi|t)\big\|_{L^2_\xi} \;\leq\; \|\langle\xi\rangle^s\hat{u}_0(\xi)\|_{L^2_\xi} \;=\; \|u_0\|_{H^s}.$$

Also,

$$\big\|\langle\xi\rangle^s\,\tfrac{\hat{u}_1(\xi)}{|\xi|}\,\sin(|\xi|t)\big\|_{L^2_\xi} \;\leq\; \big\|\tfrac{\langle\xi\rangle^s}{|\xi|}\,\hat{u}_1(\xi)\big\|_{L^2_\xi},$$

and

$$\frac{\langle\xi\rangle^s}{|\xi|} \;\approx\; \langle\xi\rangle^{s-1},$$

which implies

$$\left\|\tfrac{\langle\xi\rangle^s}{|\xi|}\,\hat{u}_1(\xi)\right\|_{L^2_\xi} \;\lesssim\; \|u_1\|_{H^{s-1}}.$$

Hence the two homogeneous parts together contribute

$$\|u(t,\cdot)\|_{H^s} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$$
$$+\left(\int_{\mathbb{R}^n}\langle\xi\rangle^{2s}\left|\int_0^t\frac{\sin((t-\tau)|\xi|)}{|\xi|}f(\tau,\xi)d\tau\right|^2 d\xi\right)^{\frac{1}{2}}$$

By Minkowski inequality,

$$\|u(t,\cdot)\|_{H^s} \lesssim \|(u_0,u_1)\|_{H^s\times H^{s-1}} + \int_0^t \left(\int_{\mathbb{R}^n} \frac{\langle\xi\rangle^{2s}}{|\xi|^2}|\hat{f}|^2 d\xi\right)^{\frac{1}{2}} d\tau$$

where $\int_0^t \left(\int_{\mathbb{R}^n} \frac{\langle\xi\rangle^{2s}}{|\xi|^2}|\hat{f}|^2 d\xi\right)^{\frac{1}{2}} d\tau \lesssim \int_0^t \|f(\tau,\cdot)\|_{H^{s-1}} d\tau$. Similar arguemnt holds for $\partial_t u$. $\square$

5.2. **Solution Formula in $n = 1, 2, 3$.**

Remark 5.7. We begin by noting that for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the function

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{\sin(t|\xi|)}{|\xi|}\hat{\varphi}(\xi)d\xi = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\cos(t|\xi|)\hat{\varphi}(\xi)d\xi.$$

For the homogeneous wave equation

$$\begin{cases} \partial_{tt}u - c^2\Delta u = 0 \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \end{cases}$$

the solution is given in Fourier space by

$$\hat{u}(t,\xi) = \cos(t|\xi|)\hat{u_0}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{u_1}(\xi).$$

Since the wave equation is linear, the solution can be constructed as the sum of two simpler solutions:

$$\text{solution with } u_0, u_1 \neq 0 \quad = \quad \underbrace{\text{solution with } u_0 = 0, u_1 \neq 0}_{=A} \quad + \quad \text{solution with } u_0 \neq 0, u_1 = 0,$$

then the solution equals $A + \partial_t A$ modulo $u_0 \leftrightarrow u_1$.

1D Case:

Remark 5.8. Recall that

$$\frac{\sin\big(t\,|\xi|\big)}{|\xi|} \;\in\; L^1(\mathbb{R}).$$

and its inverse Fourier transform is given by

$$\mathcal{F}^{-1}\Big[\frac{\sin\big(t\,|\xi|\big)}{|\xi|}\Big](x) \;=\; \frac{\sqrt{\pi}}{2}\,\chi_{(-t,t)}(x),$$

where

$$\chi_{(-t,t)}(x) \;=\; \begin{cases} 1, & \text{if } -t \le x \le t, \\ 0, & \text{otherwise.} \end{cases}$$

For $\hat{u}_1 \in \mathcal{S}(\mathbb{R})$, the product

$$\frac{\sin\big(t\,|\xi|\big)}{|\xi|}\,\hat{u}_1(\xi)$$

is still an $L^1(\mathbb{R})$ function in ξ. Hence its inverse Fourier transform is

$$\begin{aligned} \mathcal{F}^{-1}\Big[\frac{\sin\big(t\,|\xi|\big)}{|\xi|}\,\hat{u_1}(\xi)\Big] \;&=\; \frac{1}{2t}\sqrt{\frac{\pi}{2}}\,\big(\chi_{(-t,t)} * u_1\big)\,(x) \\ &= \frac{1}{2}\int_{\mathbb{R}} \chi_{(-t,t)}(x-y)u_1(y)dy \\ &= \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy. \end{aligned} \tag{5.4}$$

This implies the following formula for $\Box u = f$ in $\mathbb{R}_t \times \mathbb{R}_x$.

Theorem 5.9 (d'Alembert's Formula)**.** *For $u_0, u_1 \in \mathcal{S}(\mathbb{R})$ and $f \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}))$, the unique classical solution to $\Box u = f$ is*

$$u(t,x) = \frac{1}{2}\big(u_0(x+t) + u_0(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f(\tau, y)dyd\tau.$$

Proof. The solution u_h of the homogeneous problem

$$\begin{cases} \Box u_i = 0, \\ u_h|_{t=0} = 0, \quad \partial_t u_h|_{t=0} = 0, \end{cases}$$

is obtained from 5.4 and its time derivative. By linearity, the full solution is $u = u_h + u_i$, where u_i solves the inhomogeneous problem

$$\begin{cases} \Box u_i = f, \\ u_i|_{t=0} = 0, \quad \partial_t u_i|_{t=0} = 0. \end{cases}$$

In view of 5.4, we know that

$$\hat{u}_i(t,\xi) = \int_0^t \frac{\sin\big((t-\tau)|\xi|\big)}{|\xi|}\hat{f}(\tau,\xi)d\tau$$

which leads to

$$\begin{aligned} u_i(t,x) &= \mathcal{F}^{-1}\left(\int_0^t \frac{\sin\big((t-\tau)|\xi|\big)}{|\xi|}\hat{f}(\tau,\xi)d\tau\right) \\ &= \frac{1}{2}\int_0^t \chi_{\big(-(t-\tau), t-\tau\big)} * f d\tau \end{aligned}$$

□

Remark 5.10. An alternate approach to solve $\Box u = f$ relies on the use of a fundamental solution $u_0 \in D'(\mathbb{R}_t \times \mathbb{R}_x)$. A possible choice is

$$u_0(t,x) = \frac{1}{2}H(t-x)H(t+x),$$

where H is the Heaviside function.

3D Case:

Remark 5.11. In $n = 3$, we cannot compute

$$\mathcal{F}^{-1}\left(\frac{\sin(t|\cdot|)}{|\cdot|}\hat{u_1}\right) = \mathcal{F}^{-1}\left(\frac{\sin(t|\cdot|)}{|\cdot|}\right) * u_1$$

directly, since $\mathcal{F}^{-1}\left(\frac{\sin(t|\cdot|)}{|\cdot|}\right)$ is not a function but only a distribution in $\mathcal{S}'(\mathbb{R}^3)$.

Definition 5.12. To compute $\mathcal{F}^{-1}\left(\frac{\sin(t|\cdot|)}{|\cdot|}\right)$, we denote the 3D-sphere of radius $t > 0$ centered at $x \in \mathbb{R}^3$ by $\partial B_t(x)$ such that

$$|\partial B_t(x)| = 4\pi t^2.$$

Lemma 5.13. *It holds that*

$$\frac{1}{4\pi}\int_{\partial B_1(0)} e^{-ix\cdot\xi}ds(x) = \frac{\sin(\xi)}{|\xi|}.$$

Proof. Note that the integral is radial in ξ, since for any rotation matrix R such that $RR^T = id$, we have

$$\int_{\partial B_1(0)} e^{-ix\cdot R\xi}ds(x) = \int_{\partial B_1(0)} e^{-iR^T x\cdot\xi}ds(x).$$

Letting $y = R^T x$, we get

$$\int_{\partial B_1(0)} e^{-iy\cdot\xi} ds(y).$$

Thus, it is sufficient to take $y = (0, 0, |\xi|) \in \mathbb{R}^3$, and the integral becomes

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{-i|\xi|t\cos\theta}\sin\theta d\theta d\varphi$$

Evaluating this integral,

$$= \frac{1}{2}\frac{e^{-i|\xi|\cos\theta}}{i|\xi|}\Big|_{\theta=0}^{\theta=\pi} = \frac{\sin(|\xi|)}{|\xi|}.$$

□

Definition 5.14. Let $f \in \mathcal{S}(\mathbb{R}^3)$, for $t \geq 0$, the spherical average of f is

$$\begin{aligned}(M_t f)(x) &= \frac{1}{4\pi t^2}\int_{\partial B_t(x)} f(y) ds(y)\\ &= \frac{1}{4\pi}\int_{\partial B_1(0)} f(x - ty) ds(y).\end{aligned}$$

Remark 5.15. $M_t f$ extends to an even function in t by taking the average over $\partial B_t(x)$ for $t < 0$.

Lemma 5.16. *For $f \in \mathcal{S}(\mathbb{R}^3)$, the map $t \mapsto M_t f \in C^\infty(\mathbb{R}; \mathcal{S}\big(\mathbb{R}^3)\big)$.*

Proof. If $f \in \mathcal{S}(\mathbb{R}^3)$, then for every nonnegative integer N there is a constant $C_N > 0$ such that

$$|f(x - ty)| \leq C_N(1 + |x - ty|)^{-N}$$

On the unit sphere $|y| = 1$, we have $|x - ty| \approx |x|$ for large $|x|$. Thus

$$|f(x - ty)| \leq C'_N(t)(1 + |x|)^{-N}.$$

Integrating with respect to $y \in \partial B_1(0)$ yields

$$|M_t f(x)| \leq C''_N(t)(1 + |x|)^{-N},$$

Since $f \in \mathcal{S}(\mathbb{R}^3)$, we can differentiate under the integral to find $M_t f \in \mathcal{S}(\mathbb{R}^3)$ for fixed t. To see that $M_t f$ is smooth in t, fix x and y. Set $z = x - t\,y$. Then

$$f\big(x - (t + h)\,y\big) = f(z - h\,y).$$

By a first-order Taylor expansion around $h = 0$,

$$f(z - h\,y) \;=\; f(z) \;-\; h\,\big(y \cdot \nabla f(z)\big) \;+\; \frac{h^2}{2}\int_0^1 (1 - \tau)\sum_{j,k}\partial^2_{j,k} f\big(z - \tau h\,y\big)\,(y_j y_k)\,d\tau.$$

Hence,

$$\frac{f(z - h\,y) - f(z)}{h} \;=\; -\big(y \cdot \nabla f(z)\big) \;+\; \frac{h}{2}\int_0^1 (1 - \tau)\sum_{j,k}\partial^2_{j,k} f\big(z - \tau h\,y\big)\,(y_j y_k)\,d\tau.$$

For $h > 0$ sufficiently small, we have

$$\begin{aligned}&\left|\frac{1}{h}\Big(f\big(x - (t + h)y\big) - f(x - ty)\Big) - \nabla f(x - ty) \cdot (-y)\right|\\ &\lesssim \max_{|\alpha|=2}\sup_{B_{|t+h|}(x)} |\partial^\alpha f(z)||h| \to 0, \quad \text{as } |h| \to 0.\end{aligned}$$

Hence $\partial_t(M_t f)$ exists and is again a linear combination of integrals of the form $\nabla f(\cdot)$ against the unit sphere, which remains in $\mathcal{S}(\mathbb{R}^3)$. Repeating this argument yields all higher derivatives in t, showing

$$t \;\mapsto\; M_t f \;\in\; C^\infty\big(\mathbb{R}; \mathcal{S}(\mathbb{R}^3)\big).$$

□

Theorem 5.17 (Kirchhoff's formula)**.** *Let $n = 3$ and $f \equiv 0$. Then the unique solution to 5.1 in $C^\infty(\mathbb{R})$ with $u_0, u_1 \in \mathcal{S}(\mathbb{R}^3)$ is given by:*

$$u(t,x) = \frac{1}{4\pi t}\int_{|x-y|=|t|} u_1(y)ds(y) + \frac{\partial}{\partial t}\Big(\frac{1}{4\pi t}\int_{|x-y|=|t|} u_0(y)ds(y)\Big).$$

Proof. As in the 1D case, it suffices to look at the case $u_0 = 0$.

$$\begin{aligned}
&(2\pi)^{\frac{3}{2}}\mathcal{F}\big(tM_t(u_1)\big)(\xi)\\
&= \frac{t}{4\pi}\int_{\mathbb{R}^3} e^{-ix\xi}\int_{\partial B_1(0)} u_1(x-ty)ds(y)dx
\end{aligned}$$

let $z = x - ty$,

$$\begin{aligned}
&= \frac{t}{4\pi}\int_{\mathbb{R}^3}\int_{\partial B_1(0)} u_1(z)e^{-i(z+ty)\cdot\xi}ds(y)dz\\
&= \underbrace{\Big(\int_{\mathbb{R}^3} u_1(z)e^{-iz\cdot\xi}dz\Big)}_{(2\pi)^{\frac{3}{2}}\hat{u}_1(\xi)}\underbrace{\Big(\frac{t}{4\pi}\int_{\partial B_1(0)} e^{-ity\cdot\xi}ds(y)\Big)}_{\frac{t\sin(t|\xi|)}{t|\xi|}}.
\end{aligned}$$

Thus the distribution $v_t \in \mathcal{S}'(\mathbb{R}^2)$ is given by

$$v_t = \mathcal{F}^{-1}\big(\frac{\sin(t|\xi|)}{|\xi|}\hat{u}_1(\xi)\big) = tM_t$$

in $\mathcal{S}'(\mathbb{R}^3)$, i.e.,

$$\langle v_t, \varphi\rangle = \frac{t}{4\pi t^2}\int_{\partial B_t(0)} \varphi(x)ds(y), \quad \forall\varphi \in \mathcal{S}(\mathbb{R}^3).$$

□

2D Case: The case where $x \in \mathbb{R}^2$ can be obtained from the 3D situation using initial data which do not depend on x_3 (method of descent).

Remark 5.18. In particular, for $u_0 = 0$ and $u_1(x) = \widetilde{u}_1(x_1, x_2)$ (but not x_3), Kirchhoff's formula yields:

$$u(t,x) = \frac{1}{4\pi t}\int_{|x-y|=|t|}\widetilde{u}_1(y)ds(y) = \frac{1}{2\pi t}\int_{|x-y|=|t|, y_3-x_3\geq 0}\widetilde{u}_1(y)ds(y).$$

The upper hemisphere can be parameterized by:

$$y_3 = x_3 + \sqrt{t^2 - |(x_1,x_2)-(y_1,y_2)|^2}, \quad |(y_1,y_2)-(x_1,x_2)| \leq |t|.$$

Therefore,

$$ds = \frac{|t|}{\sqrt{t^2-|x-y|^2}}dy,$$

and

$$\begin{aligned}
u(t,x_1,x_2) = \frac{\operatorname{sgn}(t)}{2\pi}\int_{|x-y|\leq|t|}\frac{\widetilde{u}_1(y)}{\sqrt{t^2-|x-y|^2}}dy.\\
+\frac{\partial}{\partial t}\left(\frac{1}{\pi}\int_{|x-y|\leq|t|}\frac{\widetilde{u}_0(y)}{\sqrt{t^2-|x-y|^2}}dy\right).
\end{aligned}$$

This solves $\Box u = 0$ in 2D with $u|_{t=0} = u_0$ and $u_t|_{t=0} = u_1$.

5.3. **Energy Conservation and Domain of Dependence.**

Theorem 5.19 (Energy Conservation)**.** *Let $u_0, u_1 \in \mathcal{S}(\mathbb{R}^n)$. Then the solution to 5.1 with $f \equiv 0$ satisfies*

$$E(t) = \int_{\mathbb{R}^n} (\partial_t u)^2 + |\nabla u|^2 dx = E(0), \quad \forall t \in \mathbb{R}.$$

Proof. For $u_0, u_1 \in \mathcal{S}(\mathbb{R}^n)$, we have

$$u \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$$

by 5.2 for which $E(t)$ is well-defined. Moreover, 5.2 implies

$$|\xi|\hat{u}(t,\xi) = \cos(t|\xi|)|\xi|\hat{u}_0 + \sin(t|\xi|)\hat{u}_1,$$

and

$$\partial_t \hat{u}(t,\xi) = -\sin(t|\xi|)|\xi|\hat{u}_0 + \cos(t|\xi|)\hat{u}_1.$$

Squaring these identities and adding them up gives

$$\begin{aligned} |\xi|^2|\hat{u}(t,\xi)|^2 + |\partial_t \hat{u}(t,\xi)|^2 &= |\hat{u}_1|^2 + |\xi|^2|\hat{u}_0|^2 \\ &= |\xi|^2|\hat{u}(0,\xi)|^2 + |\partial_t \hat{u}(0,\xi)|^2. \end{aligned}$$

Integrating over $\xi \in \mathbb{R}^n$ and using Parseval's identity gives the result.

□

Remark 5.20. An alternative approach is to differentiate $E(t)$ with respect to time and use the PDE.

Remark 5.21. The solution formula in $n = 1, 2, 3$ shows that $\Box u = 0$ has finite speed of propagation in the sense that initial data at $y \in \mathbb{R}^n$ only influence the solution at (t, x) such that

$$|x - y| \leq t.$$

Definition 5.22. More precisely,

(a) We say that $P = (t_0, x_0) \in \mathbb{R}^{1+n}$ influences a future point $Q = (t, x)$ with $t \geq t_0$ if for every neighborhood $\Omega \subset \mathbb{R}^n$ of x_0, there exist two solutions u, v to the homogeneous wave equation such that

$$(u, \partial_t u) = (v, \partial_t v)$$

on $\{t_0\} \times (\mathbb{R}^n \backslash \Omega)$, but $(u, \partial_t u) \neq (v, \partial_t v)$ at $Q = (t, x)$.

(b) The future domain of influence of P is the set of all Q which are influenced by P.

(c) The domain of dependence of $Q \in \mathbb{R}^{1+n}$ is the set of past points P which influence Q.

Theorem 5.23. *Consider $\Box u = 0$ on $\mathbb{R}^{1+n}$ with $n = 1, 2, 3$.*

- *If $n = 1, 2$, then the domain of influence of a point (t_0, x_0) is the set:*

$$\{(t, x) : t \geq t_0, |x - x_0| \leq t - t_0\}$$

  (forward light cone).

- *If $n = 3$, it is the set:*

$$\{(t, x) : t \geq t_0, |x - x_0| = t - t_0\}.$$

Similarly, the domain of dependence of (t, x) is:

- *If $n = 1, 2$,*

$$\{(t_0, x_0) : t \leq t_0, |x - x_0| \leq t - t_0\}.$$

- *If* $n = 3$,
$$\{(t_0, x_0) : t \leq t_0, |x - x_0| = t - t_0\}.$$

Proof. Without loss of generality, let $t_0 = 0$. For $n = 1, 2$, the domain of influence follows from the solution formula, since if
$$\text{supp } (u_0, u_1) \subset \overline{B_R(x_0)},$$
then
$$\text{supp } \big(u(t, \cdot), \partial_t u(t, \cdot) \big) \subset \{x \in \mathbb{R}^n : \exists y \in \overline{B_R(x_0)}, |x - y| \leq t\}$$
for $t > 0$. Hence, if we choose $u_0 \equiv 0$ and $u_1 \geq 0$ on $\overline{B_R(x_0)}$, then
$$\text{supp } u(t, \cdot) = \{x \in \mathbb{R}^n : \exists y \in B_R(x_0) \text{ with } |x - y| \leq t\}.$$
Letting $R \to 0$ yields the result.

For $n = 3$, the solution formula involves integrals over spheres (not balls). Repeating the same argument yields the result with $|x - y| = t$. This is known as Huygens' Principle. □

5.4. **Extension of Solution Formulas.**

Lemma 5.24. *Using finite speed of propagation, we can extend the solution functions in* $n = 1, 2, 3$ *to initial data* $u_0, u_1 \in C^\infty(\Omega)$*, which are not necessarily decaying.*

Proof. Given $\varphi \in C_0^\infty(\mathbb{R}^n)$, with $\varphi \geq 0$ and such that $\varphi \neq 0$ on $[0, 1]^n$, we construct a family $\{\varphi_\alpha\}_{\alpha \in \mathbb{Z}^n} \subset C_0^\infty(\mathbb{R}^n)$ via
$$\varphi_\alpha(x) = \varphi(x - \alpha)$$
and set
$$\psi_\alpha(x) = \frac{\varphi_\alpha(x)}{\sum_{\alpha \in \mathbb{Z}^n} \varphi_\alpha(x)}.$$
Here, the sum only involves finitely many terms at each given $x \in \mathbb{R}^n$. Moreover,
$$\text{supp } \varphi_\alpha = \text{supp } \varphi + \alpha,$$
and
$$\sum_\alpha \psi_\alpha(x) = 1.$$
Thus, we write
$$u_0 = \sum_\alpha u_0 \psi_\alpha, \quad u_1 = \sum_\alpha u_1 \psi_\alpha,$$
which allows us to solve $\Box u = 0$ for $u_0, u_1 \in C^\infty(\mathbb{R}^n)$ by adding up solutions corresponding to initial data $u_0 \psi_\alpha, u_1 \psi_\alpha \in C^\infty(\mathbb{R}^n)$. Due to finite speed of propagation, this sum will only involve finitely many terms at every $x \in \mathbb{R}^n$, yielding a solution $u(t, x) \in C^\infty(\mathbb{R}^{n+1})$. □

Lemma 5.25. *Let* $n = 1$ *and* $u_0 \in C^2(\mathbb{R}), u_1 \in C^1(\mathbb{R})$*. Then, d'Alembert's formula yields a solution* $u \in C^2(\mathbb{R}^{1+1})$ *to*
$$\begin{cases} \Box u = 0, \\ u|_{t=0} = u_0, \ \ \partial_t u|_{t=0} = u_1. \end{cases}$$

Proof. Follows from the transformation $(x, t) \to (\xi, \eta)$, where $\xi = x + t$, $\eta = x - t$, in which case $\Box$ becomes $\frac{\partial^2}{\partial \xi \partial \eta}$. □

Remark 5.26. For $n = 3$, however, we have a loss of regularity, since Kirchhoff's formula only involves integrals over 2D-spheres.

Proposition 5.27. *Let $n = 3$ and $u_0 \in C^k(\mathbb{R}^3)$, $u_1 \in C^{k-1}(\mathbb{R}^3)$ with $k \geq 3$. Then Kirchhoff's formula yields a solution*

$$u \in C^{k-1}(\mathbb{R}^{1+3})$$

to

$$\begin{cases} \Box u = 0, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1. \end{cases}$$

Proof. Follows by a change of variables such that

$$\frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|x-y|=t} u_0(y)\,ds(y)\right)$$

$$\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|x|=|t|} u_0(x+ty)\,ds(y)\right)$$

and differentiate under the integral sign. □

Remark 5.28. Recall that this loss of regularity is not present if we use H^s-estimates. Question: What about uniqueness in such cases? This follows by using a localized version of the energy conservation.

Proposition 5.29. *Let $B_R(0) \subset \mathbb{R}^{n+1}$ for some $R > 0$ and*

$$\Gamma = \{(t,x) : |x| \leq R - t\}.$$

Assume that $u \in C^2(\Gamma)$ is a solution to $\Box u = 0$ such that $(u, \partial_t u) \equiv 0$ inside $\{0\} \times B_R(0) \subset \mathbb{R}^{1+n}$. Then $u(t,x) \equiv 0$ in Γ.

Proof. To prove this result, recall that for $F \in C^1(\bar{\Omega}, \mathbb{R}^n)$,

$$\int_\Omega \text{div } F\,dx = \int_{\partial\Omega} F \cdot \nu\,ds,$$

where ν is the outer unit normal vector. Using the product rule with $v \in C^1(\bar{\Omega}; \mathbb{R})$,

$$\text{div}(vF) = \nabla v \cdot F + v\text{div}\vec{F}.$$

Thus, we get the multidimensional integration by parts

$$\int_\Omega \text{div}(vF)\,dx = -\int_\Omega \nabla v \cdot F\,dx + \int_{\partial\Omega} vF \cdot \nu\,ds.$$

Without loss of generality, let $t \geq 0$ and let $\nu(x) = \frac{x}{R-t}$ be the outer unit normal vector of $B_{R-t}(0) \subset \mathbb{R}^{1+n}$. Define

$$e(t) = \frac{1}{2}\int_{|x|\leq R-t} (\partial_t u)^2 + |\nabla u|^2\,dx.$$

For $0 \leq t \leq R$, Reynolds transport theorem yields

$$\frac{d}{dt}e(t) = \int_{|x|\leq R-t} \partial_t u\,\partial_{tt} u + \nabla u \cdot \nabla\partial_t u\,dx.$$

$$-\frac{1}{2}\int_{|x|=R-t} (\partial_t u)^2 + |\nabla u|^2\,ds.$$

Recall the integrating by parts result above

$$\int_\Omega \nabla v \cdot F\,dx = -\int_\Omega \text{div}(vF)\,dx + \int_{\partial\Omega} vF \cdot \nu\,ds.$$

Let $v = \partial_t u$, and $F = \nabla u$, we have

$$\int_\Omega \nabla u \cdot \nabla\partial_t u dx = -\int_{|x|\leq R-t} \partial_t u\,\partial_{tt} u + \int_{|x|=R-t} \partial_t u \nabla u \cdot \nu ds$$

Hence,

$$\frac{d}{dt}e(t) = -\frac{1}{2}\int_{|x|=R-t} (\partial_t u)^2 + |\nabla u|^2 - 2\partial_t u \nabla u \cdot \nu ds$$

Since $2|\partial_t u \nabla u \cdot \nu| \leq |\partial_t u|^2 + |\nabla u|^2$, and $|\nu| = 1$, we obtain $\dot{e}(t) \leq 0$. Moreover, $e(0) = 0$ by assumption, so $e(t) = 0$ for all $t \geq 0$, implying

$$\partial_t u(t, \cdot) = 0, \quad \nabla u(t, \cdot) = 0, \quad \forall t \geq 0.$$

Thus, $u(t, x)$ is a constant, and since it is zero at $t = 0$, we conclude $u(t, x) = 0$. $\square$

Remark 5.30. This yields uniqueness by applying this result to $w = u_1 - u_2$, such that $\Box u_{1,2} = 0$ and

$$(u_1, \partial_t u_1) = (u_2, \partial_t u_2) \text{ at } t = 0.$$

6. The Poisson Equation

Motivation: Gauss's law of electrostatics states that

$$\operatorname{div} \vec{E} = \rho$$

where ρ is the charge density. Since $\operatorname{curl} \vec{E} = 0$, $\vec{E}$ can be written as the gradient of some potential

$$\vec{E} = -\nabla \varphi.$$

Hence, we have the Poisson equation:

$$-\Delta \varphi = \rho.$$

6.1. **Fundamental Solution.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^1$. Consider the Dirichlet boundary problem

$$\begin{cases} -\Delta u \ = \ f & \text{in } \ \Omega, \\ u \ = \ g & \text{on } \ \partial\Omega. \end{cases}$$

We want the fundamental solution u_0 satisfying

$$-\Delta u_0 \ = \ \delta_0.$$

Taking the Fourier transform of $-\Delta u_0 = \delta_0$ yields $|\xi|^2\, \widehat{u_0}(\xi) = (2\pi)^{-\frac{n}{2}}$, hence

$$\widehat{u_0}(\xi) \ = \ \frac{1}{(2\pi)^{\frac{n}{2}}\, |\xi|^2} \quad \text{for } \xi \neq 0$$

which is radial. Hence, we make the ansatz $u_0 = \psi(r)$, where $|x| = r = \sqrt{x_1^2 + \cdots + x_n^2}$.Then

$$\begin{aligned} \frac{\partial \psi}{\partial x_i} \ &= \ \psi'(r)\, \frac{x_i}{r}, \\ \frac{\partial^2 \psi}{\partial x_i^2} \ &= \ \psi''(r)\, \frac{x_i^2}{r^2} \ + \ \psi'(r)\, \Big(\frac{1}{r} \ - \ \frac{x_i^2}{r^3}\Big). \end{aligned}$$

Thus,

$$\begin{aligned} \Delta u_0 \ &= \ \psi''(r) \sum_{i=1}^{n} \frac{x_i^2}{r^2} \ + \ \psi'(r) \sum_{i=1}^{n} \Big(\frac{1}{r} \ - \ \frac{x_i^2}{r^3}\Big) \\ &= \psi''(r) \ + \ \frac{n-1}{r}\, \psi'(r), \quad \text{for } r > 0. \end{aligned}$$

Notice that $\Delta u_0 = 0$ implies the ODE

$$\psi''(r) \ + \ \frac{n-1}{r}\, \psi'(r) \ = \ 0 \ \iff \ \big(r^{\,n-1}\, \psi'(r)\big)' \ = \ 0.$$

Integrating yields

$$r^{\,n-1}\, \psi'(r) \ = \ C_1.$$

So we have $\psi'(r) = C_1\, r^{\,1-n}$ for $n \geq 2$, $C_1 \in \mathbb{R}$. A second integration gives

$$\psi(r) \ = \ \begin{cases} C_2\, \ln r + C_3, & n = 2, \\ C_2\, r^{2-n} + C_3, & n \geq 3. \end{cases}$$

Recall that for $n = 1$, we already knows $u_0(x) = \frac{1}{2}|x|$.

Theorem 6.1. *The locally integrable function $u_0 \in \mathcal{D}'(\mathbb{R}^n)$ defined by*

$$u_0(x) \ = \ \begin{cases} \dfrac{1}{2\pi}\, \ln|x|, & n = 2, \\ \dfrac{r^{\,2-n}}{(2-n)\, S_n}, & n \geq 3, \end{cases}$$

where

$$S_n \;=\; \frac{n\,\pi^{n/2}}{\Gamma\big(\frac{n}{2}+1\big)},$$

satisfies $\Delta u_0 = \delta_0$ in the sense of distributions.

Proof. To check $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$, let $K \subset \mathbb{R}^n$ be compact. Since u_0 is radial, for $n \geq 2$:

$$\int_K |u_0(x)|\,dx \;\simeq\; \int_0^R |\,\psi(r)\,|\,r^{\,n-1}\,dr \;<\; \infty.$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be any test function, then $\Delta\varphi \in \mathcal{D}(\mathbb{R}^n)$, and the support of $u_0\,\Delta\varphi$ is contained in some compact set K. Since

$$\left|\int_{\mathbb{R}^n} u_0 \Delta\varphi dx\right| \;=\; \left|\int_K u_0\Delta\varphi\;dx\right| \;\leq\; \|\Delta\varphi\|_{L^\infty}\int_K \big|u_0(x)\big|\;dx \;<\; \infty,$$

we have $u_0\,\Delta\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$. Hence,

$$\int_{\mathbb{R}^n} \Delta\varphi u_0\,dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} u_0\Delta\varphi\,dx$$

where $\Omega_\varepsilon = \mathbb{R}^n \setminus \overline{B_\varepsilon(0)}$. It remains to show

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} u_0\,\Delta\varphi\,dx \;=\; \varphi(0), \quad \forall\,\varphi \in \mathcal{D}(\mathbb{R}^n).$$

Using integration by parts twice yields

$$\int_{\Omega_\varepsilon} u_0\,\Delta\varphi\,dx \;=\; \int_{\Omega_\varepsilon} (\Delta u_0)\,\varphi\,dx \;+\; \int_{\partial B_\varepsilon(0)} \Big(u_0\,(\nabla\varphi\cdot\nu) \;-\; \varphi\,(\nabla u_0\cdot\nu)\Big)\,dS,$$

where $\nu = -\frac{x}{|x|}$ is the outward normal on $\partial B_\varepsilon(0)$. Since $\Delta u_0 = 0$ in Ω_ε, the first term vanishes. We estimate

$$\left|\int_{\partial B_\varepsilon} u_0(\nabla\varphi\cdot\nu)\,dS\right| \leq \|\nabla\varphi\|_{L^\infty}|u_0(\epsilon)||\partial B_\varepsilon| \;\to 0 \quad \text{as } \varepsilon \to 0$$

where $|\partial B_\varepsilon| = \epsilon^{n-1}S_n$. Notice that

$$\begin{aligned}\nabla u_0\cdot\nu &= \frac{du_0}{dr}(\epsilon)\frac{x}{|x|}\cdot\Big(-\frac{x}{|x|}\Big)\\ &= -\frac{du_0}{dr}(\epsilon).\end{aligned}$$

Therefore, the final integral equals

$$\begin{aligned}-\int_{\partial B_\varepsilon} \varphi(\nabla u_0\cdot\nu)\,ds &= \int_{\partial B_\varepsilon} \varphi\frac{\partial u_0}{\partial r}\,ds\\ &= \frac{\partial u_0}{\partial r}\int_{\partial B_\varepsilon} \varphi(x)\,ds.\end{aligned}$$

By the mean value theorem, $\exists\, x_\varepsilon \in \partial B_\varepsilon(0)$ such that

$$\int_{\partial B_\varepsilon} \varphi(x)\,ds = \varphi(x_\epsilon)|\partial B_\epsilon|.$$

Then,

$$\begin{aligned}\frac{\partial u_0}{\partial r}\int_{\partial B_\varepsilon} \varphi(x)\,ds &= \frac{\partial u_0}{\partial r}\varphi(x_\epsilon)|\partial B_\epsilon|\\ &= \frac{\partial u_0}{\partial r}\varphi(x_\epsilon)\epsilon^{n-1}S_n.\end{aligned}$$

One checks that

$$\frac{\partial u_0}{\partial r}(\varepsilon) = \begin{cases}\frac{1}{2\pi\varepsilon}, & n = 2\\ \frac{1}{S_n}\varepsilon^{1-n}, & n \geq 3.\end{cases}$$

Hence

$$-\int_{\partial B_\varepsilon} \varphi(\nabla u_0 \cdot \nu)\, ds = \varphi(x_\varepsilon) \cdot \frac{1}{S_n} \varepsilon^{1-n} \cdot \varepsilon^{n-1} \cdot S_n$$
$$= \varphi(x_\varepsilon) \to \varphi(0) \quad \text{as } \varepsilon \to 0.$$

□

Corollary 6.2. *For any $f \in \mathcal{D}(\mathbb{R}^n)$, the function*

$$(u_0 * f) \in C^\infty(\mathbb{R}^n)$$

is a smooth solution to $\Delta u = f$ on $\mathbb{R}^n$.

Question 6.3. How to take into account boundary conditions?

Definition 6.4. The Green's function (of 1st kind) is given by

$$G(x,y) = u_0\big(\,|x-y|\,\big) - h_x(y)$$

for $x, y \in \Omega$, $x \neq y$ where

$$\begin{cases} \Delta_y h_x = 0 & \text{on } \Omega, \\ h_x(y) = u_0(|x-y|) & \forall y \in \partial\Omega. \end{cases}$$

Hence

$$\begin{cases} \Delta_y G(x,\cdot) = \delta_x & \text{on } \Omega, \\ G(x,\cdot) = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 6.5 (Representation formula)**.** *Let $f, g \in C(\overline{\Omega})$, and $u \in C^2(\Omega)$ be a classical solution to*

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then u is given by

$$u(x) = \int_{\partial\Omega} g(y) \underbrace{\frac{\partial G}{\partial \nu}(x,y)}_{=\nabla G \cdot \nu} ds(y) + \int_\Omega G(x,y) f(y)\, dy \tag{6.1}$$

for $x \in \Omega$.

Proof. For $x \in \Omega$, let $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon(x)}$. The boundary $\partial\Omega_\varepsilon$ consists of $\partial\Omega$ plus the small sphere $\partial B_\varepsilon(x)$. Integration by parts (twice) yields

$$\int_{\Omega_\varepsilon} u_0(|x-y|) \Delta u(y)\, dy$$
$$= \int_{\Omega_\varepsilon} u(y) \underbrace{\Delta_y u_0(|x-y|)}_{=0 \text{ on } \Omega_\epsilon} dy$$
$$+ \int_{\partial\Omega} \left(u_0 \frac{\partial u}{\partial \nu} - u \frac{\partial u_0}{\partial \nu} \right) ds + \underbrace{\int_{\partial B_\varepsilon(x)} \left(u_0 \frac{\partial u}{\partial \nu} - u \frac{\partial u_0}{\partial \nu} \right) ds.}_{\to u(x) \text{ as } \epsilon \to 0, \text{ similar to the previous proof}}$$

Therefore,

$$u(x) \;=\; \int_\Omega u_0(|x-y|)\, \Delta u(y)\, dy \;-\; \int_{\partial\Omega} \Big(u_0 \,\frac{\partial u}{\partial \nu} - u \,\frac{\partial u_0}{\partial \nu} \Big)\, ds. \tag{1}$$

Since $u_0(|x-y|)$ does not vanish on $\partial\Omega$, we introduce h_x. Since $\Delta h_x = 0$ on Ω, by integration by parts

$$0 = -\int_\Omega u \Delta h_x\, dy = -\int_{\partial\Omega} \left(u \frac{\partial h_x}{\partial \nu} - h_x \frac{\partial u}{\partial \nu} \right) ds - \int_\Omega h_x \Delta u\, dy \tag{2}$$

By adding (1) and (2), we have

$$u(x) \;=\; \int_\Omega (u_0 - h_x)\,\Delta u dy \;-\; \int_{\partial\Omega} \Big[(u_0 - h_x)\,\frac{\partial u}{\partial \nu} - u\,\frac{\partial}{\partial \nu}(u_0 - h_x) \Big]\,ds.$$

On $\partial\Omega$ one has $G(x, y) = 0$ and $u(y) = g(y)$. So the boundary integral simplifies, yielding

$$u(x) = \int_\Omega G(x, y) f(y)\,dy + \int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu}(x, y) ds(y).$$

□

6.2. **Green's function for balls and the half-plane.** In general, it is difficult to obtain G, except in some simple geometries.

Theorem 6.6. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ and denote $K(x, y) = \frac{1}{\pi}\left(\frac{y}{x^2+y^2}\right)$, then with $x \in \mathbb{R}, y > 0$,*

$$\begin{aligned} u(x, y) &= \big(K(\cdot, y) * g \big)\,(x) \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{y g(x_0)}{(x - x_0)^2 + y^2}\,dx_0 \end{aligned}$$

is a smooth solution $u \in C^\infty(\mathbb{R} \times (0, \infty))$ to the Dirichlet problem on the half-space, i.e.,

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R} \times (0, \infty) \\ u(\cdot, 0) = g & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

Moreover, u takes on its boundary condition in a continuous way.

Proof. The formula itself follows by using a partial Fourier transform, see homework. □

Remark 6.7. Moreover, for $g \in \mathcal{S}(\Omega)$, taking the fourier transform of the boundary condition yields

$$\hat{u}(\xi, y) = \hat{g}(\xi) e^{-y|\xi|}.$$

Therefore, taking inverse fourier transform,

$$\begin{aligned} |u(x, y) - g(x)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \hat{g}(\xi) e^{-y|\xi|} d\xi - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \hat{g}(\xi) d\xi \right| \\ &= \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \hat{g}(\xi) \big(e^{-y|\xi|} - 1 \big) d\xi \right| \\ &\le \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}(\xi)| |e^{-y|\xi|} - 1|\,d\xi \end{aligned}$$

Since $|e^{-y|\xi|} - 1| \le 2$, using Dominated Convergence theorem,

$$\lim_{y \to 0} |u(x, y) - g(x)| = 0.$$

Hence u takes on boundary condition continuously.

Remark 6.8. Since for $y > 0$,

$$|u(x, y)| \le \frac{1}{\pi} \|g\|_{L^\infty} \int_{\mathbb{R}} \frac{y}{x^2 + y^2}\,dx < \infty,$$

we see that the solution formula extends to $g \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which yields a classical solution $u \in C^2\big(\mathbb{R} \times (0, \infty)\big)$. (This remark requires a different proof!)

Definition 6.9. The function $\widetilde{K}_H(x,y,x_0)$ is called the Poisson kernel on the half plane. It can be seen to be equal to:

$$\begin{aligned}\widetilde{K}_H(x,y,x_0) &= -\frac{\partial G_H}{\partial \nu} = -\frac{\partial G_H}{\partial y}(x,y,x_0,0)\\ &= \frac{1}{\pi}\frac{y}{(x-x_0)^2+y^2},\end{aligned}$$

where

$$G_H(x,y,x_0,y_0) = \frac{1}{4\pi}\left[\ln\big((x-x_0)^2+(y-y_0)^2\big) - \ln\big((x-x_0)^2-(y+y_0)^2\big)\right]$$

which makes the theorem above a special case of the representation formula (6.1) with $f \equiv 0$.

Remark 6.10. The Green's function G_H is obtained using the "method of images". Take the 2D fundamental solution

$$u(x,y) = \frac{1}{2\pi}\ln(\sqrt{x^2+y^2}) = \frac{1}{4\pi}\ln(x^2+y^2)$$

Then, for $y>0$, we have

$$G_{\mathbb{H}}(x,y,x_0,y_0) = \underbrace{\tau_{(x_0,y_0)}u_0(x,y)}_{\text{fundamental sol. with singularity at }(x_0,y_0)} - \underbrace{\tau_{(x_0,-y_0)}u(x,y)}_{h_x(y)}.$$

Then,

$$\Delta G = \delta_{x_0,y_0} \text{ on } \mathbb{R}\times(0,\infty), \text{ such that } G=0 \text{ on x-axis.}$$

A second important situation is the case where $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $R>0$, with $n \geq 3$.

Definition 6.11. For $x \in B_R(0)$, the reflection of x through $\partial B_R(0)$ is defined by

$$\overline{x} \;=\; \frac{R^2}{|x|^2}\,x.$$

For $x,y \in B_R(0)$, $x \neq y$ the Green's function in the ball is

$$G_B(x,y) \;=\; \begin{cases} u_0\big(|x-y|\big) \;-\; u_0\Big(\frac{|y|}{R}\,|x-\overline{y}|\Big), & y\neq 0,\\ u_0\big(|x-y|\big) \;-\; u_0(R), & y=0.\end{cases}$$

Notice that for $y \neq 0$,

$$\frac{|y|}{R}|x-\overline{y}| = \sqrt{\frac{|x|^2|y|^2}{R^2}+R^2-2x\cdot y},$$

which implies

$$G_B(x,y) \;=\; u_0\Big(\sqrt{|x|^2+|y|^2-2\,x\cdot y}\Big) \;-\; u_0\Big(\sqrt{\tfrac{|x|^2\,|y|^2}{R^2}+R^2-2\,x\cdot y}\Big), \quad y\neq 0.$$

One checks that $G_B(x,y) = G_B(y,x)$ and $G_B(x,y) \leq 0$ for all $x,y \in \overline{B_R(0)}$. Alength computation shows

$$\begin{aligned}\frac{\partial G_B}{\partial \nu} \;&=\; \nabla_y G_B\,\frac{y}{R}\\ &= \frac{R^2\,x}{R\,S_N\,|x-y|^n}\\ &= K_B(x,y)\end{aligned}$$

which is the Poisson kernel of balls.

Theorem 6.12 (Poisson formula for balls)**.** *Let $g \in C(\partial B_R(0))$. Then*

$$u(x) \;=\; \frac{R^2 - |x|^2}{R\, S_N} \int_{\partial B_R(0)} \frac{g(y)}{|x-y|^n} \,\mathrm{d}S(y) \tag{6.2}$$

defines a classical solution $u \in C^2\big(B_R(0)\big) \cap C\big(\overline{B_R(0)}\big)$ to

$$\begin{cases} \Delta u \;=\; 0, & \text{on } \Omega = B_R(0), \\ u \;=\; g, & \text{on } \partial B_R(0). \end{cases}$$

Proof. Proof that 6.2 solves the PDE with the boundary condition on $B_R(0)$ follows from the representartion formula 6.1. To show that u takes on g continuously, let $x_0 \in \partial B_R(x_0)$ and $\varepsilon > 0$. Since g is continuous, $\exists \delta > 0$ such that

$$|y - x_0| < \delta \Rightarrow |g(y) - g(x_0)| < \frac{\varepsilon}{2}.$$

Now consider $x \in B_R(0)$ such that $|x - x_0| < \frac{\delta}{2}$. Observe that our Poisson kernel satisfies

$$\int_{\partial B_R(0)} K_B(x,y) \,\mathrm{d}S(y) \;=\; 1.$$

Therefore, we have

$$|u(x) - g(x_0)| = \left| \int_{\partial B_R(x_0)} K_B(x,y)(g(y) - g(x_0))\, dS(y) \right|$$

$$\leq \int_{|y-x_0|<\delta} |K_B(x,y)||g(y) - g(x_0)|\, dS + \int_{|y-x_0|>\delta} |K_B(x,y)||g(y) - g(x_0)|\, dS$$

$$\leq \frac{\varepsilon}{2} + 2 \sup_{z \in \partial B_R} |g(z)| \cdot \frac{R^2 - |x|^2}{RS_N} \int_{|y-x_0|>\delta} \frac{dS}{|x-y|^n}.$$

The last integral is finite since

$$|x - y| \geq |y - x_0| - |x_0 - x| \geq \frac{\delta}{2}.$$

For $|x - x_0| \to 0$, we find $R^2 - |x|^2 \to 0$ since $x_0 \in \partial B_R(x_0)$, and hence

$$|u(x) - g(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $|x - x_0|$ sufficiently small. ∎

Corollary 6.13. *Let Ω be open and bounded, u be harmonic, i.e., $\Delta u = 0$ in Ω. Then $u \in C^\infty(\Omega)$.*

Proof. For $\Omega = B_R(0)$, this follows from 6.2 since $K_B(x,y)$ is C^∞ in $x \in B_R(x_0)$ for $y \in \partial B_R(x_0)$. For general open Ω, let $x \in \Omega$ then $\exists B_R(x) \subset \Omega$ such that $u \in C^\infty$ at $x \in \Omega$. ∎

Remark 6.14. By Poisson's formula, for $u \in C^2(\overline{B_R(0)})$,

$$u(x) = \frac{R^2 - |x|^2}{R\omega_n} \int_{\partial B_R(0)} \frac{u(y)}{|x-y|^n}\, dS(y).$$

Then, at $x = 0$,

$$u(0) = \frac{R}{S_n} \int_{\partial B_R(0)} \frac{u(y)}{|y|^n}\, dS(y) = \frac{1}{S_n R^{n-1}} \int_{\partial B_R} u\, dS$$

which is the average over sphere.

6.3. **Properties of harmonic functions.**

Theorem 6.15 (Mean Value Property)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C^2(\Omega)$ be harmonic. Then $\forall R > 0$ and $x \in \Omega$ such that $\overline{B_R(x)} \subset \Omega$ it holds*

$$u(x) = M_R(x) = \underbrace{\frac{1}{S_N R^{n-1}} \int_{\partial B_R(x)} u(y)\, dS(y).}_{\text{average over sphere}}$$

Moreover, if $u \geq 0$ or $u \leq 0$ resp., then $u(x) \leq M_R(x)$ or $u(x) \geq M_R(x)$.

Proof. Let $x_0 \in \Omega$ and $r = |x - x_0|$. By Gauss's Theorem, for u with $\Delta u = 0$ we have

$$0 \;=\; \int_{B_r(x_0)} \Delta u \,\mathrm{d}x \;=\; \int_{\partial B_r(x_0)} \nabla u \cdot \nu dS.$$

Parametrizing $\partial B_r(x_0)$ via $\omega = \frac{|x-x_0|}{r}$ shows

$$\int_{\partial B_r(x_0)} \nabla u \cdot \nu dS \;=\; r^{\,n-1} \int_{|w|=1} \frac{\partial u}{\partial r}(x_0 + r\, w) d\omega \;=\; 0.$$

Next, integrate in r from 0 to R and swap the integrals

$$\begin{aligned} 0 \;&=\; \int_{|w|=1} \int_0^R \frac{\partial u}{\partial r}(x_0 + r\, w) dr d\omega \\ &=\; \int_{|w|=1} u(x_0 + R\, w) - u(x_0) d\omega. \end{aligned}$$

Subsitute w with $\widetilde{w} = R\, w$, thus $|\widetilde{w}| = R$ and $\mathrm{d}\omega = \frac{1}{R^{n-1}} \mathrm{d}\widetilde{\omega}$, yielding

$$\begin{aligned} 0 \;&=\; \frac{1}{R^{n-1}} \int_{|\widetilde{w}|=R} u(x_0 + \widetilde{w}) d\widetilde{\omega} - u(x_0) \int_{|w|=1} d\widetilde{\omega} \\ &=\; \frac{1}{R^{n-1}} \int_{\partial B_R(x_0)} u(y)\, \mathrm{d}S(y) \;-\; u(x_0)\, S_N. \end{aligned}$$

Hence

$$u(x_0) \;=\; \frac{1}{S_N\, R^{n-1}} \int_{\partial B_R(x_0)} u(y) dS(y).$$

Similarly for the cases $\Delta u \geq 0$ or $\Delta u \leq 0$. □

Remark 6.16. For $n = 1$: $u'' = 0$ yields $u(x) = ax + b$, for $a, b \in \mathbb{R}$. Take an interval $I = [x_1, x_2] \subset \mathbb{R}$, then

$$\underbrace{u\left(\frac{x_1 + x_2}{2}\right)}_{\text{value at midpoint}} = \underbrace{\frac{1}{2} u(x_1) + \frac{1}{2} u(x_2)}_{\text{average over boundary}}\, .$$

Theorem 6.17 (Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set, and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $\Delta u \geq 0$ in Ω (or $\Delta u \leq 0$ in Ω, respectively). Then*

(a) *If there exists a point $x_0 \in \Omega$ such that*

$$u(x_0) = \max_{\overline{\Omega}} u \qquad \left(or\ u(x_0) = \min_{\overline{\Omega}} u \right),$$

then u is constant on Ω.

(b) *The maximum (or minimum) of u is attained on the boundary:*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \qquad \left(or\ \inf_{\Omega} u = \inf_{\partial\Omega} u \right).$$

Remark 6.18. Connected means that Ω cannot be written as the disjoint union of two nonempty open sets.

Remark 6.19. (a) is called the *strong* maximum principle, while (b) is called the *weak* maximum principle.

Proof. (a) Let $M = u(x_0) = \sup_\Omega u(x)$ and $0 < r < \operatorname{dist}(x_0, \partial\Omega)$. By the mean value property and the fact $u \leq M$,

$$\begin{aligned} M = u(x_0) &\leq \frac{1}{S_n r^{n-1}} \int_{\partial B_r(x_0)} u \, dS \\ &\leq \frac{M}{S_n r^{n-1}} |\partial B_r(x_0)| \\ &= M. \end{aligned}$$

Hence, $u = M$ on $B_r(x_0)$.

The set $A = \{x \in \Omega : u(x) = M\}$ is open, since by the argument above, for any $x \in A$, $\exists r > 0$ such that $B_r(x) \subset A$ with $u = M$ on $B_r(x)$.

Moreover, A is the preimage of $\{M\}$ under $u \in C(\overline{\Omega})$ Hence A is closed. Then $B = A^c$ is open, and since $A \neq \emptyset$, and Ω is connected, we conclude that $B = \emptyset$ and thus $A = \Omega$.

(b) Follows from (a), since if u takes on its sup inside Ω (hence maximum by continuity) $\Rightarrow u =$ constant. Similarly for $\inf u$. □

Corollary 6.20 (Uniqueness)**.** *Let $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ be solutions to*

$$\begin{cases} \Delta u = 0 \ \text{on } \Omega \\ u = g \ \text{on } \partial\Omega, \end{cases}$$

then $u_1 = u_2$ on Ω.

Proposition 6.21 (Harnack's inequality)**.** *Let $u \in C^2(\Omega)$ be harmonic and non-negative. Let $\Omega_1 \subset \Omega$ be a bounded open subset such that $\overline{\Omega_1} \subset \Omega$. Then there exists $C_1 = C(\Omega_1) > 0$ such that*

$$\sup_{\Omega_1} u(x) \leq C_1 \inf_{\Omega_1} u(x)$$

Remark 6.22. Note that C_1 does not depend on u! Hence, non-negative harmonic functions cannot oscillate too much on bounded sets.

Proof. (sketch) Let $R > 0$ and $\overline{B_R(0)} \subset \Omega$. Then by Poisson's formula, one finds:

$$\frac{R^{n-2}(R - |x|)}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n-2}(R + |x|)}{(R - |x|)^{n-1}} u(0)$$

for $|x| < R$. (exercise!) Take $r > 0$ such that $B_r(0) \subset \Omega$. Then $\exists R > r$ such that the inequality above holds. Then,

$$\sup_{x \in B_r(0) \subset B_R(0)} u(x) \leq \frac{R^{n-2}(R + r)}{(R - r)^n} u(0) = C_0 u(0)$$

and

$$\inf_{x \in B_r(0)} u(x) \geq \frac{R^{n-2}(R - r)}{(R + r)^n} u(0) = \widetilde{C_0} u(0)$$

Combining these two yields

$$\sup_{x \in B_r(0)} u(x) \leq \frac{C_0}{\widetilde{C_0}} \inf_{x \in B_r(0)} u(x).$$

Since $\overline{\Omega_1}$ is compact, we can cover it with finitely many balls to obtain the result. □

Remark 6.23. Poisson's formula can also be used to show that a bounded harmonic function defined on all of $\mathbb{R}^n$ must be constant (Liouville's Theorem).

7. Weak Solutions to Elliptic PDEs

Let $\Omega \subset \mathbb{R}^n$ be bounded and connected, with $\partial\Omega \in C^1$. We consider

$$(*) \quad \begin{cases} L(x,\nabla)u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{7.1}$$

where $L(x,\nabla)u = -\operatorname{div}(A(x)\nabla u) + b(x)\cdot\nabla u + c(x)u$. Then $A(x) = \big(a_{ij}(x)\big)_{1<i,j<n}$, $b = \big(b_i(x)\big)_{1<i<n}$, such that $a_{ij}, b_i, c \in L^\infty(\Omega)$ for all $i,j = 1,\dots,n$, and $f \in L^2(\Omega)$.

In addition, we assume that $A(x)$ is symmetric and uniformly elliptic, i.e., $\exists\alpha > 0$ s.t.

$$y^T A(x) y \geq \alpha |y|^2 \quad \forall y \in \mathbb{R}^n,\ x \in \Omega.$$

Remark 7.1. For discontinuous coefficients a_{ij}, b_i, c, there may not be a classical C^2-solution of 7.1. In general, we need a more general framework (weak solutions), elements of Sobolev spaces.

7.1. Sobolev Spaces (an introduction).

Definition 7.2. Let $\Omega \subset \mathbb{R}^n$ be open. Then

$$H^k(\Omega) := \{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega)\ \forall |\alpha| \leq k\},$$

where $\partial^\alpha u$ denotes the weak (or distributional) derivative, i.e.,

$$\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|}\langle u, \partial^\alpha \varphi\rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Remark 7.3. The alternative notation is $H^k(\Omega) \equiv W^{k,2}(\Omega)$.

Definition 7.4. Clearly, $H^k(\Omega)$ is a vector space and we endow it with the inner product:

$$\langle u, v\rangle_{H^k} := \sum_{|\alpha|\leq k} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2(\Omega)},$$

and corresponding norm

$$\|u\|_{H^k} = \sqrt{\langle u, u\rangle_{H^k}}.$$

We also set $H^0(\Omega) \equiv L^2(\Omega)$.

Definition 7.5. Let $\Omega \subset \mathbb{R}^n$ be open, and define

$$H_0^k(\Omega) := \overline{\mathcal{D}^\infty(\Omega)}^{\|\cdot\|_{H^k}},$$

i.e., the closure of $C_0^\infty(\Omega)$ with respect to the H^k-norm.

Proposition 7.6. *$H^k(\Omega)$ and $H_0^k(\Omega)$ are Hilbert spaces.*

Proof. To show that H^k is complete, let $(u_j)_{j\in\mathbb{N}} \subset H^k(\Omega)$ be a Cauchy sequence. Since

$$\|u_j\|_{H^k}^2 = \sum_{|\alpha|\leq k} \|\partial^\alpha u_j\|_{L^2}^2,$$

all $(\partial^\alpha u_j)_{j\in\mathbb{N}}$ are Cauchy in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete, there exists $u_\alpha \in L^2(\Omega)$ such that as $j \to \infty$,

$$\|\partial^\alpha u_j - u_\alpha\|_{L^2} \to 0.$$

Denote the zeroth derivative $u_{(0,\dots,0)} = u_0$, then we need to show that $\partial^\alpha u_0 = u_\alpha$. Indeed, for $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned} \langle u_\alpha, \varphi\rangle_{L^2} &= \langle \lim_{j\to\infty} \partial^\alpha u_j, \varphi\rangle_{L^2} \\ &= \lim_{j\to\infty} \langle \partial^\alpha u_j, \varphi\rangle_{L^2} \end{aligned}$$

since $\langle\cdot,\varphi\rangle$ is continuous on L^2. By definition of the weak derivative,

$$\lim_{j\to\infty} \langle \partial^\alpha u_j, \varphi\rangle_{L^2} = \lim_{j\to\infty} (-1)^{|\alpha|}\langle u_j, \partial^\alpha\varphi\rangle_{L^2}$$

$$= (-1)^{|\alpha|} \langle \lim_{j\to\infty} u_j, \partial^\alpha \varphi \rangle_{L^2}$$
$$= (-1)^{|\alpha|} \langle u_0, \partial^\alpha \varphi \rangle_{L^2}.$$

Hence, $\partial^\alpha u_0 = u_\alpha$ for all $|\alpha| \leq k$.

$H_0^k(\Omega)$ is complete. $H_0^k(\Omega)$ is a closed subspace of $H^k(\Omega)$ and hence complete. □

Remark 7.7. Note that $H^k(\Omega) \subset H^{k-1}(\Omega)$ such that and the identity map id : $H^k \to H^{k-1}$ is continuous.

Example 7.8. Consider the following examples:

(a) Let $\Omega = (-1, 1) \in \mathbb{R}$ and $u(x) = \frac{1}{2}|x|$ is in $H^1(-1, 1)$, since we already know that $u' = H(x)$, the Heaviside function. Thus,
$$u \in H^1((-1, 1)).$$
However, $u \notin H^2((-1, 1))$, since $u''(x) = \delta_0$.

(b) Let $x \in \mathbb{R}^3$ and $u(x) = \ln(|x|)$ for $x \neq 0$. Then $u \in H^1\big(B_1(0)\big)$, as one can check using spherical coordinates.

Remark 7.9. One can show that for $\Omega \subset \mathbb{R}^n$ bounded and open, $C^\infty(\Omega)$ is dense in $H^k(\Omega)$. (Lieb-Loss "Analyssis").

Remark 7.10. Elements in $H^k(\Omega)$ are equivalence classes of functions. In particular, if $u = v$ in $H^k(\Omega)$, then $u(x) = v(x)$ for all $x \in \Omega \backslash N$, where meas$(N) = 0$.

Question 7.11. We would like to make sense of
$$H_0^k(\Omega) := \{u \in H^k(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$
How shall we define this notion?

Theorem 7.12 (Trace of Sobolev Functions)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded such that $\partial\Omega \in C^1$. Then there exists a bounded linear operator*
$$T : H^1(\Omega) \to L^2(\partial\Omega)$$
called the trace operator, such that for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$, we have
$$Tu = u|_{\partial\Omega}.$$

Proof. Let $u \in H^1(\Omega) \cap C(\overline{\Omega})$. Let $x_0 \in \partial\Omega$. Assume we have "straightened" $\partial\Omega$ near x_0 so that $\partial\Omega$ becomes "$\subseteq$" $\{x_n = 0\}$ as shown in the figure below.

Let $r > 0$ and choose $\eta \in C_0^\infty(B_{x_0}(r))$ such that $0 \leq \eta \leq 1$, and $\eta \equiv 1$ on $B_{x_0}(\frac{r}{2})$. Denote
$$\Gamma = \partial\Omega \cap B_{x_0}(r), \quad B_+ = B_{x_0}(r) \cap \{x_n > 0\}.$$
Then
$$\|u\|^2_{L^2(\Gamma)} = \int_\Gamma u^2\, ds \leq \int_{\overline{B_+} \cap \{x_n = 0\}} \eta u^2\, ds.$$
Because supp$\eta \subseteq B_{x_0}(r)$,
$$\int_{\overline{B_+} \cap \{x_n = 0\}} \eta u^2\, ds = \int_{\partial B_+} \eta u^2 ds.$$
By the divergence theorem
$$\int_{\partial B_+} \eta u^2 ds = -\int_{B_+} \frac{\partial}{\partial x_n}(\eta u^2)\, dx$$
$$= -\int_{B_+} \left(u^2 \frac{\partial \eta}{\partial x_n} + 2\eta u \frac{\partial u}{\partial x_n} \right) dx$$

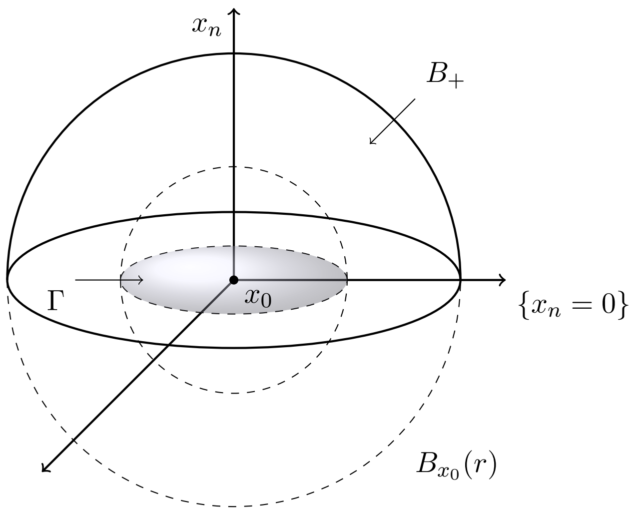

FIGURE 1. *"Straightened" $\partial\Omega$ near x_0*

$$
\begin{aligned}
&\le \left\| \frac{\partial \eta}{\partial x_n} \right\|_\infty \int_{B_+} u^2\, dx + 2\|\eta\|_\infty \int_{B_+} \underbrace{|u| \left| \frac{\partial u}{\partial x_n} \right|}_{\le |u|^2 + \left|\frac{u}{\partial x_n}\right|^2} dx \\
&\le C \int_{B_+} \left(u^2 + \left| \frac{\partial u}{\partial x_n} \right|^2 \right) dx \\
&\le C \|u\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Since $\partial\Omega \in C^1$, for each $x_0 \in \partial\Omega$ there exists an open neighborhood where $\partial\Omega$ can be straightened. Since $\partial\Omega$ is compact, it can be covered by finitely many such small open neighborhoods. Thus we can find $C_\Omega > \delta$ such that $u \in C^1(\overline{\Omega})$ yields

$$\|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} \le C_\Omega \|u\|^2_{H^1(\Omega)}.$$

Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ by standard functional analysis, we conclude that

$$\exists T : H^1(\Omega) \to L^2(\partial\Omega)$$

such that $Tu = u|_{\partial\Omega}$ for $u \in C^1(\overline{\Omega})$. □

Remark 7.13. Let $u \in H^1_0(\Omega)$. Then there exists a sequence $(u_j)_j \subset C_0^\infty(\Omega)$ such that $u_j \to u$ in $H^1(\Omega)$. If $Tu_j = u_j\big|_{\partial\Omega} \equiv 0$ for each j, then $Tu_j \to Tu$ in $L^2(\partial\Omega)$. Hence $Tu = 0$ in $L^2(\partial\Omega)$ *if and only if* $u_j\big|_{\partial\Omega} \to 0$ in $L^2(\partial\Omega)$.

Remark 7.14. If $u \in H^1(\Omega)$ and $Tu = 0$, then $u \in H^1_0(\Omega)$. Consequently,

$$H^1_0(\Omega) \;=\; \big\{ u \in H^1(\Omega) :\, Tu = 0 \big\}.$$

Corollary 7.15. *One can generalize the Divergence Theorem as the following. Let $\Omega \subset \mathbb{R}^n$ be open, bounded with $\partial\Omega \in C^1$. Let $u = (u_1, \dots, u_n) \in H^1(\Omega)$ and $v \in H^1(\Omega)$. Then,*

$$\int_\Omega v \operatorname{div}(u)\, dx = -\int_\Omega u \cdot \nabla v\, dx + \int_{\partial\Omega} (Tu \cdot \vec{\nu})\, Tv\, ds,$$

where $Tu \in L^2(\partial\Omega)$, $Tv \in L^2(\partial\Omega)$.

Theorem 7.16 (Poincaré Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega \in C^1$. Then there exists $C_P > 0$ such that*

$$\|u\|_{L^2(\Omega)} \le C_P \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1_0(\Omega).$$

Proof. Let $u \in C_0^\infty(\Omega)$. Since Ω is bounded, $\exists L > 0$ such that $\Omega \subset (-L, L)^n \subset \mathbb{R}^n$. Then, by the chain rule

$$\|u\|_{L^2(\Omega)}^2 = \int_\Omega \frac{\partial}{\partial x_1}(x_1 u^2)\, dx - \int_\Omega x_1 \frac{\partial}{\partial x_1}(u^2)\, dx.$$

By the Divergence Theorem

$$\|u\|_{L^2(\Omega)}^2 = \int_{\partial\Omega} x_1 \nu_1 u^2\, dx - 2\int_\Omega x_1 u \frac{\partial u}{\partial x_1}\, dx.$$

The first integral equals to zero since $u \in C_0^\infty(\Omega)$. Then,

$$\|u\|_{L^2(\Omega)}^2 \leq 2L\|u\|_{L^2(\Omega)} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\Omega)}.$$

Hence,

$$\begin{aligned}\|u\|_{L^2(\Omega)} &\leq 2L \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\Omega)} \\ &\leq 2L\|\nabla u\|_{L^2(\Omega)}.\end{aligned}$$

Let $u \in H_0^1(\Omega)$, then $\exists (u_j)_{j\geq 1} \in C_0^\infty(\Omega)$ such that $u_j \to u$ in $H^1(\Omega)$, then we have

$$\|u_j\|_{L^2} \to \|u\|_{L^2},$$

and

$$\|\nabla u_j\|_{L^2} \to \|\nabla u\|_{L^2}.$$

where $\|u_j\|_{L^2}^2 \leq 2L\|\nabla u_j\|_{L^2}^2, \forall j \geq 1$. Hence,

$$\|u\|_{L^2} \leq 2L\|\nabla u\|_{L^2}.$$

□

Corollary 7.17. *Let $u \in H_0^1(\Omega)$ for Ω open and bounded with $\partial\Omega \in C^1$. Then,*

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq (C_P^2 + 1)\|\nabla u\|_{L^2(\Omega)}^2,$$

and,

$$\|\nabla u\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}.$$

So, on $H_0^1(\Omega)$, the norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\nabla \cdot\|_{L^2(\Omega)}$ are equivalent.

Definition 7.18. Let $\Omega \subset \mathbb{R}^n$ be open and $k \in \mathbb{N}$. Then, denote the dual of $H_0^k(\Omega)$ as

$$H^{-k}(\Omega) = \{u : H_0^k(\Omega) \to \mathbb{R} \quad \text{linear and continuous}\}$$

and its norm

$$\|u\|_{H^{-k}(\Omega)} = \sup_{\|\varphi\|_{H_0^k(\Omega)}=1} |u(\varphi)|.$$

Remark 7.19. Note that $u \in H^{-k}(\Omega)$ is a distribution, and we use

$$u(\varphi) = \langle u, \varphi\rangle \quad \text{for } u \in H^{-k}(\Omega),\ \varphi \in H_0^k(\Omega).$$

In addition,

$$\mathcal{D}(\Omega)H^k(\Omega) \subset L^2(\Omega) \subset H^k(\Omega) \subset H^{-k}(\Omega) \subset \mathcal{D}'(\Omega).$$

7.2. **Existence of Weak Solutions.** Consider the boundary value problem with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla u\big) + b(x)\cdot\nabla u + c(x)u = f(x) & \text{in } \Omega \subseteq \mathbb{R}^n, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.2}$$

For $u \in C^2(\Omega)\cap C(\overline{\Omega})$, we can multiply the PDE by a test function $v \in C_0^\infty(\Omega)$ and integrate

$$\int_\Omega (\nabla u \cdot A\nabla v + b\cdot\nabla u\, v + cuv)\,dx = \int_\Omega fv\,dx.$$

Define

$$a(u,v) = \int_\Omega \nabla u\cdot A\nabla v + b\cdot\nabla u\, v + cuv\,dx$$

$$F(v) = \int_\Omega fv\,dx.$$

Definition 7.20. We define

(a) The identity $a(u,v) = F(v)$ for all $v \in H_0^1(\Omega)$ solved for $u \in H_0^1(\Omega)$ is called the weak formulation of 7.2.

(b) A function $u \in H_0^1(\Omega)$ is called a weak solution of 7.2 if

$$a(u,v) = F(v), \quad \forall v \in H_0^1(\Omega).$$

Remark 7.21. If $u \in C^2(\Omega)\cap C(\overline{\Omega})$ is a classical solution of 7.2, then u is also a weak solution of 7.2.

Remark 7.22. Conversely, if a weak solution u is such that $u \in C^2(\Omega)\cap C(\overline{\Omega})$, then for $v \in C_0^\infty(\Omega)$ dense in $H_0^1(\Omega)$, we integrate by parts in $a(u,v) = F(v)$to obtain

$$\int_\Omega \Big(-\operatorname{div}(A(x)\nabla u) + b(x)\cdot\nabla u + c(x)u - f(x)\Big)v\,dx = 0$$

$\forall v \in C_0^\infty(\Omega)$, which implies

$$-\operatorname{div}\big(A(x)\nabla u\big) + b(x)\cdot\nabla u + c(x)u = f(x)$$

almost everywhere in Ω.

In the easiest case $A(x) = \mathrm{Id}$, $b(x) = 0$, $c(x) = 1$, then $a(u,v) = F(v)$ is equivalent to

$$\int_\Omega \nabla u\cdot\nabla v + uv\,dx = \int_\Omega fv\,dx, \quad \forall v \in H_0^1(\Omega),$$

where $\int_\Omega \nabla u\cdot\nabla v + uv\,dx = \langle u,v\rangle_{H^1(\Omega)}$.

Theorem 7.23 (Riesz's representation theorem)**.** *Let H be a Hilbert space and $F \in H'$ a linear continuous functional, then there exists a unique $u \in H$ such that*

$$\langle u,v\rangle_H = F(v), \quad \forall v \in H.$$

Moreover,

$$\|F\|_{H'} = \|u\|_H.$$

Corollary 7.24. *Let $f \in L^2(\Omega)$, then there exists a weak solution $u \in H_0^1(\Omega)$ to the case where $A(x) = Id$, $b(x) = 0$, $c(x) = 1$, i.e.*

$$\begin{cases} -\Delta u + u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Notice that
$$|F(v)| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H^1}.$$
Hence,
$$F \in H^{-1}(\Omega) \equiv (H_0^1)'.$$
Then Riesz's representation theorem applies. $\square$

Corollary 7.25. *Let $f \in L^2(\Omega)$, then there exists a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ on Ω such that $Tu = 0$ on $\partial\Omega$.*

Proof. The expression $a(u, v) = \langle \nabla u, \nabla v\rangle_{L^2}$ defines an inner product on $H_0^1(\Omega)$, and Poincaré's inequality implies that $H_0^1(\Omega)$ is complete under the induced norm
$$\|u\|_{L^2} = \sqrt{\langle \nabla u, \nabla u\rangle_{L^2}}.$$
Riesz's representation theorem applies to yield a unique $u \in H_0^1(\Omega)$ such that $\langle \nabla u, \nabla v\rangle_{L^2} = F(v)$, for all $v \in H_0^1(\Omega)$. $\square$

Remark 7.26. Here we have used that $\|\nabla u\|_{L^2} = 0$, then u is constant, hence $u = 0$ since $u \in H_0^1(\Omega)$. This is used to show the positive-definiteness of the inner product $a(u, v)$.

Definition 7.27. Let H be a Hilbert space and let $a : H \times H \to \mathbb{R}$ be a bilinear form. Then

(a) a is continuous if there exists $K_1 > 0$ such that
$$|a(u, v)| \leq K_1\|u\|_H\|v\|_H \quad \forall u, v \in H.$$

(b) a is coercive if there exists $K_2 > 0$ such that
$$a(u, u) \geq K_2\|u\|_H^2 \quad \forall u \in H.$$

Theorem 7.28 (Lax-Milgram Lemma)**.** *Let $a : H \times H \to \mathbb{R}$ be bilinear, continuous and coercive, and let $F \in H'$. Then there exists a unique $u \in H$ such that*
$$a(u, v) = F(v) \quad \forall v \in H,$$
and
$$\|u\|_H \leq \frac{1}{K_2}\|F\|_{H'}.$$

Proof. Let $w \in H$, then $a(w, \cdot) : H \to \mathbb{R}$ is linear and continuous. Hence, $a(w, \cdot) \in H'$. Riesz representation theorem implies that there exists a unique $S(w) \in H$ such that
$$\langle S(w), v\rangle = a(w, v) \quad \forall v \in H.$$
Moreover, by Riesz's representation theorem
$$\|S(w)\|_H = \|a(w, \cdot)\|_{H'}$$
and by continuity of a,
$$\|a(w, \cdot)\|_{H'} = \sup_{v\neq 0} \frac{|a(w, v)|}{\|v\|_H} \leq K_1\|w\|_H.$$
This yields a linear, continuous map $S : H \to H$, and we want to show it can be inverted. If so, then we are done, since if $g \in H$ is the unique solution to $\langle g, v\rangle = F(v)$, $\forall v \in H$, then
$$a\big(S^{-1}(g), v\big) = \langle g, v\rangle = F(v) \quad \forall v \in H,$$
i.e., $u = S^{-1}(g)$ is the desired solution.

To show that there exists a linear, continuous S^{-1}, we first prove:

Step 1: S is injective.

By coercivity of a, $\forall u \in H$

$$K_2\|u\|^2 \leq a(u,u) = \langle S(u), u\rangle \leq \|S(u)\|\|u\|$$

yields

$$K_2\|u\| \leq \|S(u)\|.$$

Hence $S(u) = 0$ implies $u = 0$, so S is injective. We can define $S^{-1} : \operatorname{ran} S \to H$, which is linear and continuous, since

$$\|S^{-1}(v)\| \leq \frac{1}{K_2}\|v\| \quad \forall v \in \operatorname{ran} S.$$

Step 2: $\operatorname{ran}(S)$ is closed.

Let $(v_n)_n \subset \operatorname{ran}(S)$ such that $\lim_{n\to\infty} v_n = v$. Then $\exists u_n \in H$ with $v_n = S(u_n)$. By the coercivity estimate from step one, $(u_n)_{n\geq 1}$ is Cauchy, and H is complete, hence $(u_n)_{n\geq 1}$ converges in H to some w. Then by continuity of S, as $k \to \infty$,

$$S(u_n) \to S(w).$$

Hence,

$$v = S(w) \in \operatorname{ran}(S).$$

Step 3: $\operatorname{ran}(S) = H$.

Assume by contradiction that $\operatorname{ran}(S) \neq H$. Then since $\operatorname{ran}(S)$ is closed,

$$H = \operatorname{ran}(S) \oplus (\operatorname{ran}(S))^{\perp},$$

with $(\operatorname{ran}(S))^{\perp} \neq \{0\}$. Let $z \in (\operatorname{ran}(S))^{\perp}$, $z \neq 0$. Then

$$\langle S(v), z\rangle = 0 \quad \forall v \in H.$$

By the definition of S,

$$0 = \langle S(z), z\rangle = a(z,z) \geq K_2\|z\|^2$$

yields $z = 0$. Contradiction. Hence $\operatorname{ran}(S) = H$. Thus, $S : H \to H$ is invertible.

Step 4: Uniqueness.

Let $u_1, u_2 \in H$ be two solutions to

$$a(u,v) = F(v) \quad \forall v \in H.$$

Then

$$a(u_1 - u_2, v) = 0 \quad \forall v \in H.$$

Taking $v = u_1 - u_2$, we obtain

$$K_2\|u_1 - u_2\|^2 \leq a(u_1 - u_2, u_1 - u_2) = 0.$$

Hence, $u_1 = u_2$. □

Lemma 7.29 (Young's Inequality)**.** *Let $x, y > 0$, $\delta > 0$, and $p, q \in (0, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$xy \leq \frac{\delta}{p}x^p + \frac{\delta^{-\frac{q}{p}}}{q}y^q.$$

Proof. We find for $\alpha, \beta > 0$,

$$\begin{aligned}\alpha\beta &= e^{\ln\alpha + \ln\beta}\\ &= \exp\Big(\frac{1}{p}\ln\alpha^p + \frac{1}{q}\ln\beta^q\Big).\end{aligned}$$

By convexity of $x \mapsto e^x$,

$$\exp\Big(\frac{1}{p}\ln\alpha^p + \frac{1}{q}\ln\beta^q\Big) \leq \frac{1}{p}e^{\ln\alpha^p} + \frac{1}{q}e^{\ln\beta^q}$$

$$= \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q.$$

Taking $\alpha = \delta^{\frac{1}{p}} x, \beta = \delta^{-\frac{1}{p}} y$ yields the desired inequality. $\square$

Theorem 7.30. *Let $a_{ij} \in L^\infty(\Omega), b_i \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$, and $A = (a_{ij})_{i,j}$ be symmetric and uniformly elliptic. Assume further that $c(x) \geq c_0 > 0$, $\forall x \in \Omega$, and $f \in L^2(\Omega)$. Moreover, assume either*

(i) *$b = (b_i) \equiv 0$, and $c_0 = 0$, or*

(ii) *$4\alpha c_0 > b_0^2$, where $b_0 = \|b\|_{L^\infty}$, and α is the ellipticity constant.*

Then there exists a unique weak solution $u \in H_0^1(\Omega)$ to

$$a(u, v) = F(v) \quad \forall v \in H_0^1(\Omega),$$

satisfying

$$\|u\|_{H_0^1} \leq C\|f\|_{L^2},$$

where

$$C = \frac{1 + {C_P}^2}{\alpha - \frac{{b_0}^2}{4c_0}},$$

where $C_P > 0$ is the Poincaré constant.

Proof. We need to show that

$$a(u, v) = \int_\Omega \nabla u \cdot A\nabla v + (b \cdot \nabla u)v + cuv\, dx$$

satisfies the conditions of the Lax-Milgram Lemma.

For continuity, we estimate

$$|a(u, v)| \leq \|\nabla u\|_{L^2}\|A\|_{L^\infty}\|\nabla v\|_{L^2} + (\|b\|_{L^\infty} + \|c\|_{L^\infty})\|u\|_{H^1}\|v\|_{L^2}$$
$$\leq K_1\|u\|_{H^1}\|v\|_{H^1}$$

For coercivity, by uniform ellipticity of A,

$$a(u, u) \geq \int_\Omega \alpha\nabla u \cdot \nabla u - b_0|u||\nabla u| + c_0u^2\, dx.$$

Apply Young's inequality for $y = |\nabla u|$ and $x = |u|$, with $p = q = 2$, yields

$$a(u, u) \geq \int_\Omega \Big(\alpha - \frac{{b_0}^2}{2\delta}\Big)|\nabla u|^2 + \Big(c_0 - \frac{\delta}{2}{b_0}^2\Big)u^2\, dx.$$

Therefore

(i) If $b \equiv 0, c_0 > 0$, we set $b_0 = 0, c_0 = 0$, hence

$$a(u, u) \geq \alpha \int_\Omega |\nabla u|^2\, dx.$$

(ii) If $4\alpha c_0 > {b_0}^2$, we choose $\delta = \frac{2c_0}{b_0}$, to obtain

$$a(u, u) \geq c_1 \int_\Omega |\nabla u|^2\, dx$$

with

$$c_1 = \alpha - \frac{b_0^2}{4c_0} > 0.$$

By Poincaré inequality, $\|\cdot\|_{H^1} \simeq \|\nabla\cdot\|_{L^2}$, hence:

$$a(u,u) \geq K_2\|u\|^2_{H^1},$$

with

$$K_2 = \frac{c_1}{1+{C_p}^2}.$$

Thus, Lax-Milgram applies to yield a unique weak solution $u \in H^1_0(\Omega)$. □

Remark 7.31. $\|u\|_{H^1_0} \leq C\|f\|_{L^2}$ shows continuous dependence of u on f.

7.3. Extensions and Variations.

7.3.1. *More general right-hand sides:* The theory directly applies to distributions $f \in H^{-1}(\Omega)$, since

$$F(v) = \langle f, v\rangle$$

which is the duality bracket between H^{-1} and $H^1_0(\Omega)$. Therefore, we have

$$|F(v)| \leq \|f\|_{H^{-1}}\|v\|_{H^1_0},$$

yielding

$$\|F\|_{H^{-1}} \leq \|f\|_{H^{-1}}$$

which allows us to apply Lax–Milgram lemma.

7.3.2. *Inhomogeneous Dirichlet Problem.* We consider:

$$\begin{cases} Lu = -\mathrm{div}(A\nabla u) + b\cdot\nabla u + cu = f, & \text{on } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$, $g \in H^1(\partial\Omega)$, then Tg is defined on $\partial\Omega$, and $Tu = Tg$ on $\partial\Omega$. We want to find a weak solution $u \in H^1(\Omega)$ such that

$$(u-g) \in H^1_0(\Omega),$$

i.e., $Tu = Tg$ on $\partial\Omega$. To do so, set $w = u - g$, rewrite the BVP in the form:

$$\begin{cases} Lw = Lu - Lg = f - Lg, & \text{in } \Omega, \\ Tw = 0, & \text{on } \partial\Omega. \end{cases}$$

The corresponding weak formulation is

$$a(w,v) = F(v) - a(g,v), \quad \forall v \in H^1_0(\Omega).$$

Let

$$G(v) = F(v) - a(g,v).$$

We can apply Lax-Milgram Lemma, provided $G(v)$ is continuous in $H^{-1}(\Omega)$. This follows since F is continuous (like previously), and

$$|G(v)| \leq \|f\|_{L^2}\|v\|_{H^1} + K_1\|g\|_{H^1}\|v\|_{H^1},$$

yielding

$$\|G\|_{H^{-1}} \leq \|f\|_{L^2} + K_1\|g\|_{H^1}.$$

Corollary 7.32. *Under the same assumptions a_{ij}, b_i, c and $f \in L^2$ as before, there exists a unique weak solution $u \in H^1(\Omega)$ to the inhomogeneous Dirichlet problem such that*

$$\|u\|_{H^1} \leq C(\|f\|_{L^2} + \|g\|_{H^1})$$

Proof. Since $G \in H^{-1}(\Omega) = (H^1_0(\Omega))'$, the Lax-Milgram lemma implies there exists a unique $w \in H^1_0(\Omega)$ such that

$$a(w+g,v) = G(v) + a(g,v) = F(v).$$

So $u = w + g \in H^1(\Omega)$ is the solution with $u = g$ on $\partial\Omega$.

□

7.3.3. *Neumann Boundary Conditions.* We seek a weak formulation of:

$$\begin{cases} Lu = f & \text{on } \Omega \\ (A\nabla u)\cdot\nu = g & \text{on } \partial\Omega \end{cases} \tag{7.3}$$

where $L = -\mathrm{div}(A\nabla\cdot) + b\cdot\nabla + c$ and ν is the outer normal unit vector on $\partial\Omega$. We multiply by $v \in H^1(\Omega)$ and integrate by parts using divergence theorem yielding

$$\begin{aligned} a(u,v) &= \int_\Omega fv\,dx + \int_{\partial\Omega} (A\nabla u)\cdot\nu v\,dS, \\ &= \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,dS. \end{aligned}$$

Let $\widetilde{F}(v) = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,dS$. Hence, we seek a solution $u \in H^1(\Omega)$ such that

$$a(u,v) = \widetilde{F}(v), \quad \forall v \in H^1(\Omega).$$

Lemma 7.33. *Let $A \in C^1(\Omega)\cap C(\overline{\Omega})$, $b, c, f \in C(\Omega)$, and $g \in C(\partial\Omega)$. Then, every classical solution u to 7.3 is a weak solution and a weak solution $u \in H^1(\Omega)$ such that $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ is a classical solution.*

Proof. Exercise! □

Remark 7.34. For $u \in H^1(\Omega)$, we cannot use Poincaré inequality to show coercivity, hence we need to impose more restrictions.

Theorem 7.35. *Let $a_{ij}, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $A = (a_{ij})$ be uniformly elliptic. Furthermore, we assume that $b \equiv 0$ and that*

$$c(x) \geq c_0 > 0 \quad \forall x \in \Omega.$$

Then for any $g \in H^1(\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ such that

$$a(u,v) = \widetilde{F}(v), \quad \forall v \in H^1(\Omega),$$

and

$$\|u\|_{H^1} \leq C_1(\|f\|_{L^2} + C_2\|g\|_{H^1})$$

where $C_1^{-1} = \min(\alpha, c_0)$.

Proof. Continuity of $a(u,v)$ follows as in the previous cases. To show coercivity, notice

$$\begin{aligned} a(u,u) &= \int_\Omega \alpha|\nabla u|^2 + c_0u^2\,d\Omega \\ &\geq \min(\alpha, c_0)\|u\|_{H^1}^2. \end{aligned}$$

Since

$$\widetilde{F}(v) = F(v) + \langle Tg, Tv\rangle_{L^2(\partial\Omega)}$$

with $F \in H^{-1}(\Omega)$. We have $\widetilde{F} \in H^1(\Omega)$ and the Lax-Milgram lemma yields existence and uniqueness. □

Remark 7.36. If $c(x) = 0$ for all $x \in \Omega$, we do not get coercivity. Indeed, this case requires additional conditions:

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{on } \Omega \\ (A\nabla u)\cdot\nu = g & \text{on } \partial\Omega. \end{cases} \tag{7.4}$$

Integrating the PDE yields

$$\begin{aligned} \int_\Omega f\,dx &= -\int_\Omega \mathrm{div}(A\nabla u)\,dx \\ &= -\int_{\partial\Omega} (A\nabla u)\cdot\nu\,dS \end{aligned}$$

$$= -\int_{\partial\Omega} g\, dS.$$

Also note that if u is a solution of 7.4, then so is $\widetilde{u} = u + \text{constant}$.

7.4. **Regularity of Weak Solutions.** Motivation: Let u be a classical solution to $\Delta u = f$ on $\mathbb{R}^n$ with $f \in L^2(\mathbb{R}^n)$, then

$$(\Delta u) \in L^2(\mathbb{R}^n).$$

Indeed, all second order derivatives of u are controlled by the L^2-norm of f, since

$$\int_{\mathbb{R}^n} f^2\, dx = \int_{\mathbb{R}^n} (\Delta u)^2\, dx = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \partial_{ii} u\, \partial_{jj} u\, dx$$

by integration by parts

$$= -\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \partial_{iij} u\, \partial_j u\, dx$$

by integration by parts again

$$= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \partial_{ij} u\, \partial_{ji} u\, dx$$

$$= \int_{\mathbb{R}^n} |D^2 u|^2\, dx$$

where $D^2 u$ is the Hessian matrix. This can be generalized to weak solutions $u \in H^1$, but the proof is rather technical (see, e.g., [2, Sec. 6.3]). The result is:

Theorem 7.37 (Interior Regularity)**.** *Let $a_{ij} \in C^{k+1}(\Omega)$, $b_i, c \in C^k(\Omega)$ and $f \in H^k(\Omega)$ for $k \in \mathbb{N}_0$. Let $u \in H^1(\Omega)$ be a weak solution to*

$$Lu = f \quad \text{on } \Omega,$$

and denote by Ω_1 a domain such that $\overline{\Omega_1} \subset \Omega$. Then $u \in H^{k+2}(\Omega_1)$ and $\exists C > 0$ such that

$$\|u\|_{H^{k+2}(\Omega_1)} \le C(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Remark 7.38. In other words, u is twice more regular (in an H^k sense) than f inside Ω. Note that the regularity of g on $\partial\Omega$ does not play a role here. Therefore, singularities on $\partial\Omega$ do not propogate inside of Ω.

Theorem 7.39 (Regularity up to the boundary)**.** *Assume the same conditions as before, and in addition let $\partial\Omega \in C^{k+2}$ and $g \in H^{k+2}(\Omega)$. Then if u is a weak solution to*

$$\begin{cases} Lu = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then $u \in H^{k+2}(\Omega)$. Moreover,

$$\|u\|_{H^{k+2}} \le C\left(\|f\|_{H^k} + \|g\|_{H^{k+2}}\right).$$

Question 7.40. What about classical solutions?

Theorem 7.41 (Sobolev embedding)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega \in C^1$, and assume that $k - \frac{n}{2} > m$ for $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then*

$$H^k(\Omega) \hookrightarrow C^m(\overline{\Omega}),$$

i.e. $\exists C > 0$ such that

$$\|u\|_{C^m(\overline{\Omega})} \le C\|u\|_{H^k(\Omega)}.$$

Proof. (sketch) We will prove the case $k=1$, $m=0$. Let $u \in C_0^\infty(\Omega)$ and $R>0$ such that $\Omega \subset B_R(x_0)$ for all $x_0 \in \Omega$. Then

$$|u(x_0)| = |u(x_0 + R\vec{\nu}) - u(x_0)|$$

because $\operatorname{supp} u \subset \Omega \subset B_R(x_0)$, $u(x_0 + R\vec{\nu}) = 0$. Hence,

$$|u(x_0)| = \left| \int_0^R \frac{d}{dr} u(x_0 + r\vec{\nu})\, dr \right|.$$

Integrating over $\vec{\nu} \in \partial B_1(0)$ yields:

$$|\partial B_1(0)||u(x_0)| \le \int_0^R \int_{\partial B_1(0)} |\nabla u(x_0 + r\vec{\nu}) \cdot \vec{\nu}|\, dS\, dr$$

$$= \int_{B_R(x_0)} |\nabla u(x)| \frac{dx}{|x - x_0|^{n-1}}$$

By Cauchy–Schwarz:

$$|\partial B_1(0)||u(x_0)| \le \|\nabla u\|_{L^2} \left(\int_{B_R(x_0)} \frac{dx}{|x - x_0|^{2(n-1)}} \right)^{1/2}$$

This integral is finite if $2(n-1) < n$. Then, for $n<2$,

$$\|u\|_{C(\overline{\Omega})} = \sup_{\Omega} |u(x_0)| \le C\|\nabla u\|_{L^2(\Omega)}$$

$$\le C\|u\|_{H^1}.$$

Approximating $u \in H^1(\Omega)$ by a sequence $(u_j) \in C_0^\infty(\Omega)$ yields the result for $k=1$, $m = 0 \Rightarrow n < 2$.
The general case can be proved similarly. ☐

For $k > \frac{n}{2}$ and assuming the conditions of the previous theorem, we get

$$u \in H^{k+2}(\Omega) \hookrightarrow C^2(\overline{\Omega}).$$

Then u is indeed a classical solution to $Lu = f$ on Ω with $u = g$ on $\partial\Omega$. In addition, if all coefficients in L are in $C^\infty(\Omega)$, and if $f \in C^\infty(\overline{\Omega})$, then

$$u \in C^\infty(\overline{\Omega_1}) \quad \text{for } \overline{\Omega_1} \subset \Omega.$$

Moreover, if $\partial\Omega \in C^\infty$, we get $u \in C^\infty(\overline{\Omega})$. The main example being harmonic functions on $B_R(0)$.

Theorem 7.42 (Rellich-Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega \in C^1$. Let $k \in \mathbb{N}$, $m \in \mathbb{N}_0$ such that $k > m$. Then*

$$H^k(\Omega) \hookrightarrow H^m(\Omega) \ \text{ compactly,}$$

i.e., bounded sets in $H^k(\Omega)$ are pre-compact in $H^m(\Omega)$. Then, if $(u_j) \subset H^k(\Omega)$ is bounded, there exists a subsequence $(u_{j_n})_{n\ge1}$ such that $u_{j_n} \xrightarrow{n\to\infty} u$ in $H^m(\Omega)$ provided $k > m$.

Proof. Evans' book. ☐

7.5. **Generalized Maximum Principle.** We consider elliptic PDEs of the form:

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b(x)\cdot\nabla u + c(x)u$$

which is not necessarily in divergence form.

Remark 7.43. If $a_{ij} \in C^1(\Omega)$, $\forall i,j = 1,\dots,n$, we can write

$$Lu = -\mathrm{div}(A\nabla u) + \tilde{b}\cdot\nabla u + cu$$

where $\tilde{b}(u) = b(x) - \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j}$.

Theorem 7.44 (Weak Maximum Principle for $c \equiv 0$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $a_{ij}, b_i \in L^\infty(\Omega)$, $\forall i,j = 1,\dots,n$, and let $A(x) = (a_{ij})$ be symmetric and uniformly elliptic. Let $c \equiv 0$ and $u \in C^2(\Omega)\cap C(\overline{\Omega})$ satisfy*

$$Lu \le 0 \quad \text{in } \Omega \quad (\text{resp. } Lu \ge 0 \text{ in } \Omega),$$

then

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \quad (\text{resp. } \inf_{\Omega} u = \inf_{\partial\Omega} u).$$

Proof. Step 1: Assume $Lu < 0$ in Ω and $\exists x_0 \in \Omega$ such that

$$u(x_0) = \sup_{\Omega} u.$$

Then $\nabla u(x_0) = 0$, and $D^2u(x_0) = B = B^T$ is negative definite. We will show that this implies $Lu(x_0) > 0$, hence a contradiction. Since $A = (a_{ij})$ is symmetric and positive definite, there exists an orthogonal matrix $S \in \mathbb{R}^{n\times n}$ such that

$$SAS^T = SAS^{-1} = \mathrm{diag}(\lambda_1,\dots,\lambda_n), \quad \lambda_j > 0, \quad \forall j = 1,\dots,n.$$

We compute (recall that $c \equiv 0$ and $\nabla u(x_0) = 0$):

$$\begin{aligned} Lu(x_0) &= -\sum_{i,j=1}^{n} a_{ij}(x_0)\underbrace{\frac{\partial^2 u}{\partial x_i\partial x_j}(x_0)}_{=b_{ij}=b_{ji}} \\ &= -\sum_{i=1}^{n}(AB^T)_{ii} \\ &= -\mathrm{tr}(AB^T) \\ &= \mathrm{tr}(AB) \end{aligned}$$

where $B = D^2u(x_0)$ is negative definite, all diagonal elements β_i of SBS^T satisfy $\beta_i < 0$, so we have

$$\begin{aligned} Lu(x_0) &= -\mathrm{tr}(AB) = -\mathrm{tr}(S^{-1}SAB) \\ &= -\mathrm{tr}(SABS^{-1}) \\ &= -\mathrm{tr}(SAS^{-1}BS^{-1}) \\ &= -\sum_{i=1}^{n}\lambda_i\beta_i \ge 0 \end{aligned}$$

a contradiction.

Step 2:Assume $Lu \le 0$, and let

$$u_\varepsilon(x) := u(x) + \varepsilon e^{\lambda x_1}, \quad \text{with } \varepsilon, \lambda > 0.$$

Since $y^T Ay \ge \alpha|y|^2$, we obtain for $y = (1,0,\dots,0) = e_1$,

$$a_{11}(x) \ge \alpha > 0, \quad \forall \alpha \in \Omega.$$

Moreover, since $b \in L^\infty(\Omega)$, $|b(x)| \le b_0$ for some $b_0 > 0$ and all $x \in \Omega$,

$$\begin{aligned} L(u_\varepsilon) &= L(u) + \varepsilon L(e^{\lambda x_1}) \le \varepsilon L(e^{\lambda x_1}) \\ &= \varepsilon(-\lambda^2 a_{11} + \lambda b_1) e^{\lambda x_1} \\ &\le \lambda\varepsilon(-\lambda\alpha + b_0)e^{\lambda x_1} < 0 \end{aligned}$$

for $\lambda > \frac{b_0}{\alpha}$. By step 1,

$$\sup_\Omega (u + \varepsilon e^{\lambda x_1}) = \sup_{\partial\Omega}(u + \varepsilon e^{\lambda x_1}),$$

and letting $\varepsilon \to 0$ yields the result. □

Corollary 7.45. *Impose the same assumptions as before and let $c \ge 0$. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$Lu \le 0 \quad (\textit{resp. } Lu \ge 0)$$

on Ω, then

$$\sup_\Omega u = \max\left\{0, \sup_{\partial\Omega} u\right\} \quad (\textit{resp. } \inf_\Omega u \ge \min\left\{0, \inf_{\partial\Omega} u\right\}).$$

Proof. Let $Lu \le 0$ on Ω and denote

$$\Omega_+ := \{x \in \Omega : u(x) > 0\}$$

If $\Omega_+ = \emptyset$, then $u(x) \le 0$ on Ω. Otherwise, we get on Ω_+

$$-\sum a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i \frac{\partial u}{\partial x_i} \le -cu \le 0.$$

By the previous theorem (and since Ω_+ is open), u takes on its supremum on

$$\partial\Omega_+ = \partial\Omega \cup \{x : u(x) = 0\}.$$

Hence for $x_0 \in \partial\Omega_+$, $u(x_0) = \sup_\Omega u$ or $u(x_0) = 0$. □

Remark 7.46. There is also a strong maximum principle which implies that for $c \equiv 0$, and $Lu < 0$, if u takes its maximum on Ω, then u is constant.

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